OBAFEMI AWOLOWO UNIVERSITY, ILE-IFE, NIGERIA
DEPARTMENT OF MATHEMATICS
Rain Semester Examination, 2021/2022 Academic Session
MTH 202 - Mathematical Methods II

Date: 27th June, 2023. **Type 4** **Time Allowed : $2\frac{1}{4}$ Hours**

Instructions: Attempt all questions. Use HB pencil ONLY. Write and shade your **question type, Names, Registration Number** and the **correct option** in your OMR sheet.

1. The matrix $A = \begin{pmatrix} -5 & 2 & 3 \\ -5 & \lambda & 3 \\ 1 & 1 & 7 \end{pmatrix}$ is invertible if

   (A) $\lambda \neq -2$, (C) $\lambda \neq \pm 2$.
   (B) $\lambda \neq 2$, (D) $\lambda = 2$

2. Let $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ and $r = |\vec{r}|$. Find $\nabla^2(-15r^{-1})$.

   (A) 0 (C) $-15r^{-1}$
   (B) $\vec{0}$ (D) $15r^{-3}\vec{r}$

3. Let f be the bilinear form on $\mathbb{R}^2$ defined by

   $$f[(x_1,x_2),(y_1,y_2)] = 2x_1y_1 - 3x_1y_2 + 4x_2y_2.$$

   The matrix A of f in the basis $\{\alpha = (1,0), \beta = (1,1)\}$ is

   (A) $\begin{pmatrix} 2 & -1 \\ 2 & 3 \end{pmatrix}$
   (B) $\begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$.
   (C) $\begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}$
   (D) $\begin{pmatrix} 1 & -2 \\ 3 & 2 \end{pmatrix}$

4. The image of the linear map $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ defined by $T(a,b,c) = (a,a)$ is

   (A) $\langle (1,1) \rangle$.
   (B) $\langle (1,0) \rangle$.
   (C) $\langle (0,1) \rangle$.
   (D) $\langle (0,0) \rangle$.

5. The principal argument of $\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) \Big/ (-1-i)$ is

   (A) π
   (B) $-\frac{13}{12}\pi$
   (C) $-\frac{\pi}{4}$
   (D) $-\frac{11}{12}\pi$

6. Let $\{u,v\}$ be a basis for $\mathbb{R}^2$ over $\mathbb{R}$. Which of the following is true?

   (A) None of the above.
   (B) $\alpha u + \beta v = 0$ for nonzero scalars α, β.
   (C) $u = \alpha v$ for some scalars β.
   (D) For any $w \in \mathbb{R}^2$, there exists nonzero scalars α, β such that $w = \alpha u + \beta v$.

7. The quadratic form corresponding to

   $$\begin{pmatrix} 4 & -5 & 7 \\ -5 & -6 & 8 \\ 7 & 8 & -9 \end{pmatrix}$$

   is

   (A) $q(x) = q(x_1,x_2,x_3) = 4x_1^2 - 10x_1x_2 - 6x_2^2 + 14x_1x_3 + 16x_2x_3 - 9x_3^2$
   (B) $q(x) = q(x_1,x_2,x_3) = 4x_1^2 + 10x_1x_2 - 6x_2^2 - 14x_1x_3 - 16x_2x_3 - 9x_3^2$
   (C) $q(x) = q(x_1,x_2,x_3) = 4x_1^2 - \frac{5}{2}x_1x_2 - 6x_2^2 + \frac{5}{2}x_1x_3 + 4x_2x_3 - 9x_3^2$
   (D) $q(x) = q(x_1,x_2,x_3) = 4x_1^2 + 10x_1x_2 + 6x_2^2 + 14x_1x_3 + 16x_2x_3 + 9x_3^2$

8. Given the matrix $A = [a_{ij}]_{mn}$, where k is any real number. What is the order of the matrix kA ?

(A) $km \times n$

(B) $m \times kn$

(C) $m \times n$

(D) $km \times kn$

9. Obtain the scale factors for a parabolic coordinate (u, v, ϕ) system defined by the transformation equations $x = uv\cos\phi, y = uv\sin\phi, z = \frac{1}{2}(u^2 - v^2)$ (where $u \geq 0$, $v \geq 0$, $0 \leq \phi < 2\pi$.)

(A) $h_u = u^2 + v^2$, $h_v = u^2 + v^2$, $h_\phi = uv$

(B) $h_u = \sqrt{u^2 + v^2}$, $h_v = \sqrt{u^2 + v^2}$, $h_\phi = uv$

(C) $h_u = u^2 + v^2$, $h_v = u^2 + v^2$, $h_\phi = 1$

(D) $h_u = \sqrt{u^2 + v^2}$, $h_v = \sqrt{u^2 + v^2}$, $h_\phi = 1$

10. The sum $(1 + i\sqrt{3})^8 - (1 + i\sqrt{3})^8$ equals

$(1+i\sqrt{3}) - (1+i\sqrt{3})^2$

$(1+i\sqrt{3} - 1 + i\sqrt{3})^2$

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(A) -256

(B) -512

(C) -128

(D) -1024

11. Which of the following set of vectors in $\mathbb{R}^2$ is linearly dependent ?

(A) $\{(1,3),(2,5)\}$

(B) $\{(1,0),(0,1)\}$

(C) $\{(10,6),(5,3)\}$

(D) $\{(0,-2),(-2,0)\}$

12. Find the Jacobian of the transformation equations $x = uv\cos\phi, y = uv\sin\phi, z = \frac{1}{2}(u^2 - v^2)$ (where $u \geq 0$, $v \geq 0$, $0 \leq \phi < 2\pi$).

(A) $(u^2 + v^2)$

(B) uv

(C) $(u^2 - v^2)$

(D) $uv(u^2 + v^2)$

13. The locus of points in the complex plane for which, given $z \in \mathbb{C}$, $\text{Arg}\left(\frac{z-1}{z+1}\right) = \frac{\pi}{4}$, is

(A) a straight line passing through $(-1, 0)$.

(B) a circle with centre $(0, 1)$ and radius $\sqrt{2}$.

(C) a straight line passing through $(0, 1)$.

(D) a circle with centre $(1, 0)$ and radius $\sqrt{2}$

14. Which of the following vectors is not linearly dependent on $(-4, 14, 2) \in \mathbb{R}^3$?

(A) $(-2, 7, 1)$

(B) $(4, -14, -2)$

(C) $(-1, 7, 3)$

(D) $(2, -7, -1)$

15. Find the value of $2x + y$ if

$$x + iy = \begin{vmatrix} 4i & i^3 & 2i \\ 1 & 3i^2 & 4 \\ 5 & -3 & i \end{vmatrix}, \text{ where } i^2 = -1.$$

(A) 74

(B) 16

(C) 29

(D) 31

16. Let differentiable vectors $\vec{A}(x,y,z,t)$ and $\vec{B}(x,y,z,t)$ satisfy equations

$$\nabla \cdot \vec{A} = 0, \quad \nabla \times \vec{A} = -\frac{\partial}{\partial t}\vec{B}, \quad \text{and} \quad \nabla \times \vec{B} = \frac{\partial}{\partial t}\vec{A}.$$

Which of the following is true?

(A) $\nabla^2 \vec{A} = 0$

(B) $\nabla^2 \vec{A} = \vec{0}$

(C) $\nabla^2 \vec{A} = \frac{\partial}{\partial t}\vec{A}$

(D) $\nabla^2 \vec{A} = \frac{\partial^2}{\partial t^2}\vec{A}$

17. Suppose $r = |\vec{r}|$ and $\vec{r} = \vec{i}x + \vec{j}y + \vec{k}z$. Find $\nabla \cdot (r^{-3}\vec{r})$.

(A) $-3r^{-2}\vec{r}$

(B) 0

(C) $-3r^{-4}\vec{r}$

(D) $\vec{0}$

18. The principal argument of $-1 - i$ is

(A) $\frac{\pi}{4}$

(B) $-\frac{\pi}{4}$

(C) $-\frac{3\pi}{4}$

(D) $\frac{3\pi}{4}$

19. Let the scalar function $\Phi = \cos(x^2 z)$ and $\nabla\Phi = -\vec{H}\sin(x^2 z)$. Find vector $\vec{H}$.

(A) $2xz\vec{j} + x^2\vec{k}$ (C) $2xz\vec{i} + x^2\vec{k}$

(B) $2xz\vec{i} + x^2\vec{j}$ (D) $2xz\vec{k} + x^2\vec{i}$

20. Find the curl of the vector $\vec{V} = r\cos\theta\vec{e}_r - r\sin\theta\vec{e}_\theta$ in a spherical coordinate (r,θ,ϕ) system defined by $x = r\sin\theta\cos\phi, y = r\sin\theta\sin\phi, z = r\cos\theta$ (where $r \geq 0$, $0 \leq \theta \leq \pi, 0 \leq \phi < 2\pi$.)

(A) $-\frac{\sin\theta}{r}\vec{e}_r$ (C) $r\sin\theta\vec{e}_\theta$

(B) $-\frac{1}{r}\vec{e}_\phi$ (D) $-\sin\theta\vec{e}_\phi$

21. Which of the following maps $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is not linear ?

(A) $T(x,y) = (x^2, y)$.

(B) $T(x,y) = (0, y)$.

(C) $T(x,y) = (x, 0)$.

(D) $T(x,y) = (x - y, 2y)$.

22. The matrix $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$ is associated with a linear map

(A) $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$

(B) $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^1$

(C) $T : \mathbb{R}^1 \longrightarrow \mathbb{R}^3$

(D) $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$

23. Let $\mathbb{Q}$ be set of rational numbers, which of the following vectors does not belong to the linear vector space $\mathbb{Q}^3$ over $\mathbb{R}$

(A) $(-1, 0, -2)$

(B) $(0, 1, 0)$

(C) $(\sqrt{2}, \sqrt{3}, \sqrt{5})$

(D) $(0, 0, 0)$

24. Suppose differentiable vector $\vec{F}(x,y,z,t) = \vec{G}e^{i\beta(t - \vec{u}\cdot\vec{r})}$, where $\vec{G}$, $\vec{u}$ are constant vectors, β is a constant scalar, and $\vec{r} = \vec{i}x + \vec{j}y + \vec{k}z$. Find $\nabla \times \vec{F}$.

(A) $i\beta\vec{F} \times \vec{u}$ (C) $i\beta\vec{u} \times \vec{F}$

(B) $i\vec{F} \times \vec{u}$ (D) $i\vec{u} \times \vec{F}$

25. If the dimension of a linear vector space V is n; $n \in \mathbb{N}$. Which of the following is not necessarily true ?

(A) The basis of V has n elements.

(B) Any $n+1$-elements of V are linearly dependent.

(C) dimension of any basis of V is n.

(D) Any subspace of V has dimension n

26. Determine the gradient of the scalar $\Phi = r\cos\theta + 2z$ in a cylindrical coordinate (r,θ,z) system defined by $x = r\cos\theta, y = r\sin\theta, z = z$ (where $r \geq 0, 0 \leq \theta \leq \pi$, $-\infty < z < \infty$.)

(A) $\cos\theta\vec{e}_r - \sin\theta\vec{e}_\theta + 2\vec{e}_z$.

(B) $\cos\theta\vec{e}_r + \sin\theta\vec{e}_\theta + 2\vec{e}_z$

(C) $\cos\theta\vec{e}_r + \sin\theta\vec{e}_\theta - 2\vec{e}_z$,

(D) $\sin\theta\vec{e}_r - \sin\theta\vec{e}_\theta + 2\vec{e}_z$

27. Find the matrix associated with the linear map $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ given by $T(x,y) = (2x - y, x + 3y, -x)$; with basis $\{(1,0),(0,1)\}$ for $\mathbb{R}^2$ and basis $\{(0,0,1),(0,1,0),(1,0,0)\}$ for $\mathbb{R}^3$.

(A) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ (C) $\begin{pmatrix} -1 & 0 \\ 1 & 3 \\ 2 & -1 \end{pmatrix}$

(B) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 3 \end{pmatrix}$ (D) $\begin{pmatrix} 2 & -1 \\ 1 & 3 \\ -1 & 0 \end{pmatrix}$

28. The eigenvalues of the matrix

$$\begin{pmatrix} 1 & 0 & 2 \\ -2 & 2 & 4 \\ 0 & 0 & 3 \end{pmatrix}$$

are

(A) $-3, -2$ and -1

(B) $-1, 2$ and 3

(C) $1, 2$ and 3

(D) $1, -2$ and -3

29. Which of the following statements is / are **true** about linear maps?
(I) They preserve only addition.
(II) They preserve only scalar multiplication.
(III) They preserve both addition and multiplication.

(A) II only
(B) I only
(C) I and II.
(D) III only

30. If

$$\triangle = \begin{vmatrix} ax & x^2 & 1 \\ by & y^2 & 1 \\ cz & z^2 & 1 \end{vmatrix}, \quad \triangle_1 = \begin{vmatrix} a & b & c \\ x & y & z \\ zy & zx & xy \end{vmatrix}$$

then,

(A) $\triangle_1 \neq \triangle$
(B) $\triangle + \triangle_1 = 0$
(C) $\triangle_1 = \triangle$
(D) None of the above

31. Let $z = \cos\theta + i\sin\theta$. Then $2(1+z)^{-1}$ simplifies as

(A) $i\tan\frac{\theta}{2}$
(B) $1 + i\cot\frac{\theta}{2}$
(C) $1 - i\tan\frac{\theta}{2}$
(D) $-i\cot\frac{\theta}{2}$

$2(1+\cos\theta+i\sin\theta)^{-1}$
$\frac{2}{(1+\cos\theta)+i\sin\theta}$
$\frac{2(1+\cos\theta-i\sin\theta)}{(1+\cos\theta)^2+(\sin\theta)^2}$
$\frac{2+2\cos\theta-2i\sin\theta}{1+2\cos\theta+\cos^2\theta+\sin^2\theta}$
$\frac{2+2\cos\theta-2i\sin\theta}{2+2\cos\theta}$
$1-\frac{i\sin\theta}{1+\cos\theta}$

32. For what values of a and b are the matrices

$$\begin{pmatrix} a & 2 \\ b & 1 \end{pmatrix} \text{ and } \begin{pmatrix} -5 & -2 \\ 9 & 5 \end{pmatrix}$$

said to be similar ?

(A) $a = 0;\ b = 3$
(B) $a = -2;\ b = -3$
(C) $a = -1;\ b = 3$
(D) $a = 1;\ b = 3$

33. Which of the following is a subspace of $\mathbb{R}^3$?

(A) $\{(a,b,c) : a \leq b \leq c\}$
(B) $\{(a,b,c) : b = a^2\}$
(C) $\{(a,b,c) : ab = 0\}$
(D) $\{(a,b,c) : a = 2b = 3c\}$

34. Let ω_κ, $\kappa = 0,1,2,\ldots,n-1$, be the nth roots of unity. Which of the following is false?

(A) Product of all the roots is -1.
(B) $\sum_{\kappa=0}^{n-1} \omega_\kappa = 0$
(C) They are in geometric progression.
(D) One imaginary root is the conjugate of one other.

35. If $A = \begin{pmatrix} 1 & 4 \\ 5 & 5 \end{pmatrix}$ and $P = \begin{pmatrix} -2 & 2 \\ 1 & 5 \end{pmatrix}$ is the invertible matrix for which the matrix A is similar to another matrix B. Then, the matrix B is

(A) $\begin{pmatrix} -1 & -4 \\ 0 & 7 \end{pmatrix}$
(B) $\begin{pmatrix} -1 & 4 \\ 0 & 7 \end{pmatrix}$
(C) $\begin{pmatrix} 1 & -4 \\ 0 & 7 \end{pmatrix}$
(D) $\begin{pmatrix} 1 & 4 \\ 0 & 7 \end{pmatrix}$

36. Given that the eigenvectors for the matrix

$$A = \begin{pmatrix} 5 & 4 & 2 \\ 4 & 5 & 2 \\ 2 & 2 & 2 \end{pmatrix}$$

are $v_1 = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$, $v_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ and $v_3 = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$ respectively, the Eigenvalues of A are respectively

(A) 1, 1 and 10
(B) 10, 1 and 1
(C) 1, 2 and 3
(D) −1, 1 and 10

37. If the rank and nullity of matrix

$$A = \begin{pmatrix} 1 & 3 \\ 0 & -2 \\ 5 & -1 \\ -2 & 3 \end{pmatrix}$$

are x and y respectively, evaluate $\frac{1}{2}x + y$.

(A) 3

(B) 2

(C) 1

(D) 4

38. Find the divergence of the vector $\vec{B} = r^2\vec{e}_1 + r\sin\theta\vec{e}_2$ in a spherical coordinate (r, θ, ϕ) system defined by $x = r\sin\theta\cos\phi, y = r\sin\theta\sin\phi, z = r\cos\theta$ (where $r \geq 0$, $0 \leq \theta \leq \pi$, $0 \leq \phi < 2\pi$.)

(A) $4r + 2\cos\theta$

(B) $2r + \cos\theta$

(C) $3r + \cos\theta$

(D) $3r - \cos\theta$

39. Calculate the kernel of the linear map $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(a,b) = (a-b, a+b)$.

(A) $\{(0,0)\}$

(B) $\{(1,1)\}$

(C) $\{(1,0)\}$

(D) $\{(0,1)\}$

40. The following statements are true for any 2 similar matrices except

(A) 2 similar matrices have the same characteristic polynomial.

(B) 2 similar matrices are invertible with the same determinant.

(C) 2 similar matrices have the same eigenvalues and eigenvectors.

(D) 2 similar matrices have the same rank.

$2(1+\cos\theta+i\sin\theta)^{-1}$

$\frac{2}{(1+\cos\theta)+(i\sin\theta)}$

$\frac{2[(1+\cos\theta)-i\sin\theta]}{(1+\cos\theta)^2+(i\sin\theta)^2}$

$\frac{2[1+\cos\theta-i\sin\theta]}{1+2\cos\theta+\cos^2\theta+i^2\sin^2\theta}$

$\frac{2[1+\cos\theta+i\sin\theta]}{1+2\cos\theta+\cos^2\theta-\sin^2\theta}$

$1+\cos^2\theta = -\sin^2\theta$

$\sin^2\theta = 1-\cos^2\theta$

$\frac{2[1+\cos\theta+i\sin\theta]}{2+2\cos\theta}$

$\frac{1+\cos\theta+i\sin\theta}{1+\cos\theta} = 1+\frac{i\sin\theta}{1+\cos\theta}$

$\frac{[1+\cos\theta]+[i\sin\theta]}{(1+\cos\theta)^2} \times (1-\cos\theta)^2$

$(1+\cos\theta)(1-\cos\theta)$