LIMITS OF REAL-VALUED FUNCTIONS OF REAL VARIABLE X

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1. DEFINITION:

Let a function $f : [a, b] \longrightarrow \mathbb{R}$ be given and let $c \in [a, b]$. A real number L is said to be a limit of f at c

$$if, \textbf{given any } \epsilon > 0, \textbf{there exists a } \delta > 0 \ \textbf{ such that}$$

$$0 < |x - c| < \delta(\epsilon), \ \text{ implies } \ |f(x) - L| < \epsilon.$$

1.1. Remarks.

(A.) The inequality $0 < |x - c|$ is equivalent to saying $x \neq c$. As such, in this definition, it is immaterial whether f is defined at c or not. **In any case** (that is whether f is defined at c or not), we exclude c from consideration in the determination of the limit.

(B.) If L **is a limit of f at c**,
then we also
(a.) say that f **converges** to L at c.
(b.)

$$L = \lim_{x \longrightarrow c} f(x)$$

(c.) say that f (x) **approaches** L as x **approaches** c (either from left or right).
(d.) use the symbolism

$$f \longrightarrow L \ \ as \ \ x \longrightarrow c$$

(e.) The definition is essentially the **BASIC TOOL** for showing that a **given function f** has a limit L at c. However, as useful as the above definition is, it does not directly help to find the limit; since it only shows whether a **given value L** is the limit of f at c or it is not. Consequently, the following questions naturally come to mind:

What then helps to or how we find right L that would serve as the limit of f at c, so that we don't necessary have to show that the L is correct by using the above definition?

The answer is **MATHEMATICAL METHODS** and these are introduced in the next section. But before, we go that, let us take some specific examples. Here we go.

1.2. **Example.** Find the following limits:
(a.)

$$\lim_{x \longrightarrow c} k.$$

(b.)

$$\lim_{x \longrightarrow c} x.$$

(c.) Determine a condition on $|x-1|$ that will assure that: $|x^2-1| < \frac{1}{2}$.

Solution: We are going to use the definition above.

(a.) We note first f(x) = k (a constant). As such as x approaches c, f(x) continues to be k. So we suspect that

$$\lim_{x \longrightarrow c} k = k.$$

To show this, we let $\epsilon > 0$ be arbitrary, and go ahead to find a $\delta(\epsilon) > 0$, such that

$$0 < |x-c| < \delta(\epsilon) \text{ implies } |f(x) - k| < \epsilon.$$

Now choosing $\delta = \epsilon$, we have that

$$0 < |x-c| < \epsilon \text{ implies } |f(x) - k| = |k-k| = 0 < \epsilon.$$

Hence

$$\lim_{x \longrightarrow c} k = k.$$

(b.) We observe first that as x approaches c, f(x) approaches c. As such we suspect therefore that

$$\lim_{x \longrightarrow c} x = c.$$

To show this, we let $\epsilon > 0$ be arbitrary, and go ahead to find a $\delta(\epsilon) > 0$, such that

$$0 < |x-c| < \delta(\epsilon) \text{ implies } |x-c| < \epsilon.$$

Clearly, if we choose $\delta = \epsilon$, we have that

$$0 < |x-c| < \epsilon \text{ implies } |x-c| < \epsilon.$$

Hence by the definition above, we have that

$$\lim_{x \longrightarrow c} x = c.$$

(c.) Let $\epsilon > 0$ be arbitrary. Suppose that $|x-c| < \delta = 1$. Then

$$|x| - |c| \leq |x-c| < 1.$$

So that

$$|x| < 1 + |c|.$$

Now

$$\begin{aligned}|x^2-1| =& |(x-1)(x+1)| \\ =& |x-1||x+1| \\ \leq& |x-1|\left(|x|+1\right) \\ \leq& |x-1|\left(2+|c|\right) \\ <& \frac{1}{2} \quad \left(If\ \ |x-1| < \frac{\frac{1}{2}}{2+|c|}\right).\end{aligned}$$

Therefore, we have two conditions on $|x-1|$:

$$|x-c| < 1$$

and

$$|x-c| < \frac{1}{4+2|c|}$$

So choosing

$$\delta(1/2) := Min\left\{1, \frac{1}{4+2|c|}\right\},$$

we obtain the required condition

$$|x-c| < \delta(1/2) := Min\left\{1, \frac{1}{4+2|c|}\right\}.$$

EXERCISES

Use either the $\epsilon - \delta$ definition to show that

(d.)

$$\lim_{x\longrightarrow 2} x^2 + 4x = 12,$$

(e.)

$$\lim_{x\longrightarrow 3} \frac{x^2-3x}{x+3} = 2,$$

(f.)

$$\lim_{x\longrightarrow 0} \frac{x^2}{|x|} = 0.$$

(g.) Determine a condition on $|x-1|$ that will assure that: $|x^2-1| < 10^{-3}$.

(e.) Determine a condition on $|x-1|$ that will assure that: $|x^3-1| < 1/n$.

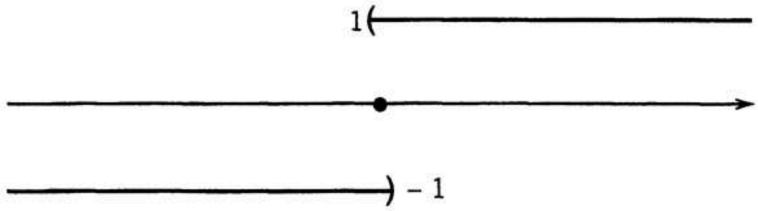

FIGURE 1. The Signum function

1.3. **Right-hand Limits and Left-Hand Limits.** There are times when a function f may not possess a limit at a point c, yet a limit does exist when the function is restricted to an interval on one side of the point c. For example, consider the figure above which represents the signum function sgn defined by

$$sgn(x) = \begin{cases} 1 & for\ x > 0, \\ 0 & for\ x = 0. \\ -1 & for\ x < 0. \end{cases}$$

has no limit at $c = 0$ (we shall know why later). However, if we restrict the signum function to the interval $(0, 1)$, the resulting function has a limit of 1 at $c = 0$. Similarly, if we restrict the signum function to the interval $(1, 0)$, the resulting function has a limit of -1 at $c = 0$. These are elementary examples of right-hand and left-hand limits at $c = 0$.

DEFINTTIONS: Right-hand Limits and Left-Hand Limits
Let $f : [a, b] \longrightarrow R$ and $c \in [a, b]$.

(a.) If $x \in (c, b)$, then we say that $L \in \mathbb{R}$ is a **right-hand limit** of f at c and we write
$$\lim_{x \longrightarrow c+} f(x) = L$$
if given any $\epsilon > 0$ there exists a $\delta(\epsilon) > 0$ such that for all $x \in [a, b]$ with $0 < x - c < \delta$, then $|f(x) - L| < \epsilon$.

(b.) If $x \in (a, c)$, then we say that $L \in \mathbb{R}$ is a **left-hand limit** of f at c and we write
$$\lim_{x \longrightarrow c-} f(x) = L$$
if given any $\epsilon > 0$ there exists a $\delta(\epsilon) > 0$ such that for all $x \in [a, b]$ with $0 < c - x < \delta$, then $|f(x) - L| < \epsilon$.

Remarks to Definitions of Right-hand Limits and Left-Hand Limits
(a.) The two limits
$$\lim_{x \longrightarrow c+} f(x) = L$$

and

$$\lim_{x \longrightarrow c-} f(x) = L$$

are called one-sided limits of f at c;

(b.) It is possible that neither one-sided limit may exist;
(c.) One of them may exist without the other existing;
(d.) They may both exist and be different;
(e.) They may both exist and be equal; and
(f.) In light of our discussion so far, it is in order to call

$$L = \lim_{x \longrightarrow c} f(x)$$

two-sided limit.

IMPORTANT NOTE:
Here, we use the word **EXIST** to mean **finite**, that is whenever the value of sided-limit is **a real number**.

1.4. Connection Between (TWO-sided) limit of f at c and ONE-sided limits of f at c.

The following result (we wont prove) gives the connection between (TWO-sided) limit of f at c and ONE-sided limits of f at c

Theorem 1.1. *Let $f : [a, b] \longrightarrow \mathbb{R}$, and let $c \in [a, b]$. Then*

$$\lim_{x \longrightarrow c} f(x) = L$$

if and only if

$$\lim_{x \longrightarrow c+} f(x) = L = \lim_{x \longrightarrow c-} f(x).$$

Remark 1.2. The theorem 1.1 simply says that one can show that the limit

$$\lim_{x \longrightarrow c} f(x)$$

EXISTS; that is

$$\lim_{x \longrightarrow c} f(x) = L, \quad L \in \mathbb{R},$$

BY SHOWING THAT **ALL** THE FOLLOW CONDITIONS HOLD:
(a.) limit

$$\lim_{x \longrightarrow c+} f(x)$$

EXISTS; that is

$$\lim_{x \longrightarrow c} f(x) = l_1, \quad l_1 \in \mathbb{R},$$

(b.) limit

$$\lim_{x \longrightarrow c-} f(x)$$

EXISTS; that is

$$\lim_{x \longrightarrow c} f(x) = l_2, \quad l_2 \in \mathbb{R},$$

(c.)

$$l_1 = l_2 (= L.)$$

Example 1.3.

(a.) Consider the signum function sgn defined by

$$sgn(x) = \begin{cases} 1 & for\ x > 0, \\ 0 & for\ x = 0. \\ -1 & for\ x < 0. \end{cases}$$

Does the limit

$$\lim_{x \longrightarrow 0} sgn(x)$$

exist?

(b.) Consider the absolute value function $|.|$ defined by

$$|x| = \begin{cases} x & for\ x > 0, \\ 0 & for\ x = 0. \\ -x & for\ x < 0. \end{cases}$$

Does the limit

$$\lim_{x \longrightarrow 0} |x|$$

exist?

SOLUTION TO EXAMPLE 1.3

(a.) It is easy to that (from figure 1), as x approaches 0 from values greater than 0 (as $x \longrightarrow 0+$), $sgn(x)$ tends (approaches) 1. Hence,

$$\lim_{x \longrightarrow 0+} sgn(x) = 1.$$

Clearly, as x approaches 0 from values less than 0 (as $x \longrightarrow 0-$), $sgn(x)$ tends (approaches) -1. Hence,

$$\lim_{x \longrightarrow 0-} sgn(x) = -1.$$

Yielding that

$$\lim_{x \longrightarrow 0+} sgn(x) = 1 \neq -1 = \lim_{x \longrightarrow 0-} sgn(x).$$

As such NOT ALL the conditions in Remark 1.2 are satisfied. Specifically, it fails the third (equality) condition. Thus

$$\lim_{x \longrightarrow 0} sgn(x)$$

does not exist.

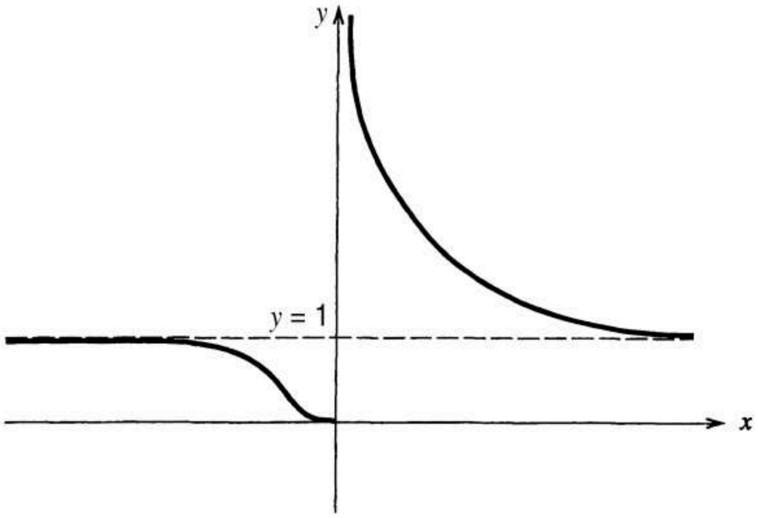

FIGURE 2. Graph of $f(x) = e^{1/x}$, $(x \neq 0)$

(b.) It is easy to that as x approaches 0 from values greater than 0 (as $x \longrightarrow 0+$), $|x|$ tends (approaches) 0. Hence,

$$\lim_{x \longrightarrow 0+} |x| = 0.$$

Clearly, as x approaches 0 from values less than 0 (as $x \longrightarrow 0-$), $|x|$ tends (approaches) -1. Hence,

$$\lim_{x \longrightarrow 0-} |x| = 0.$$

Yielding that

$$\lim_{x \longrightarrow 0+} |x| = 0 = \lim_{x \longrightarrow 0-} |x|.$$

As such ALL the conditions in Remark 1.2 are satisfied. Thus

$$\lim_{x \longrightarrow 0} sgn(x)$$

exists. As a matter of fact

$$\lim_{x \longrightarrow 0} sgn(x) = 0.$$

EXERCISES

Use the Fig 2 and Fig. 3 to answer the following questions:
(c.) Show that the limit

$$\lim_{x \longrightarrow 0+} e^{1/x} = \infty; \quad \lim_{x \longrightarrow 0-} e^{1/x} = 0.$$

Does the limit

$$\lim_{x \longrightarrow 0-} e^{1/x}$$

exist? Justify your answer.

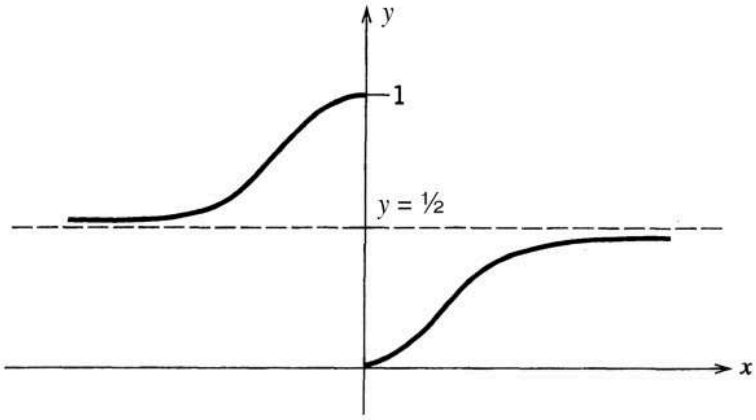

FIGURE 3. Graph of $f(x) = 1/\left(e^{1/x}+1\right)$ $(x \neq 0)$

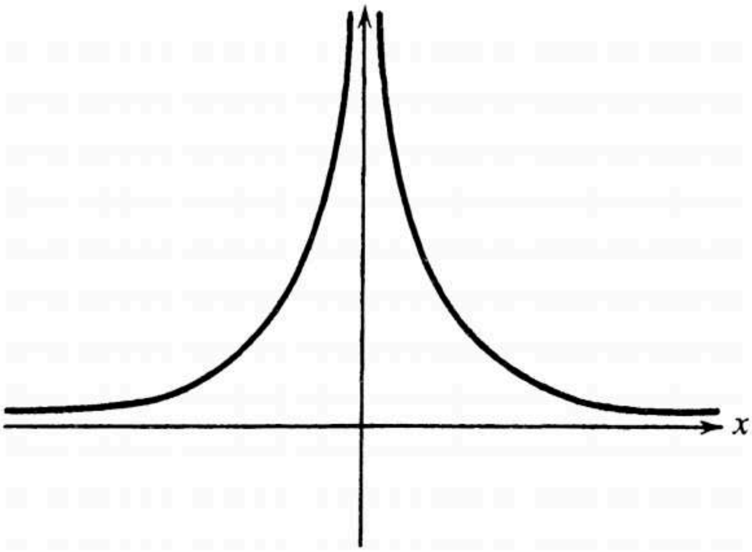

FIGURE 4. Graph of $f(x) = 1/x^2$ $(x \neq 0)$

(d.) Show that

$$\lim_{x \longrightarrow 0+} \frac{1}{e^{1/x}+1} = 0 \quad ; \lim_{x \longrightarrow 0-} \frac{1}{e^{1/x}+1} = 1$$

Does the limit

$$\lim_{x \longrightarrow 0-} \frac{1}{e^{1/x}+1}$$

exist? Justify your answer.

1.5. **Infinite Limits.** The function

$$f(x) = \frac{1}{x^2} \quad x \neq 0$$

(see Figure 4)

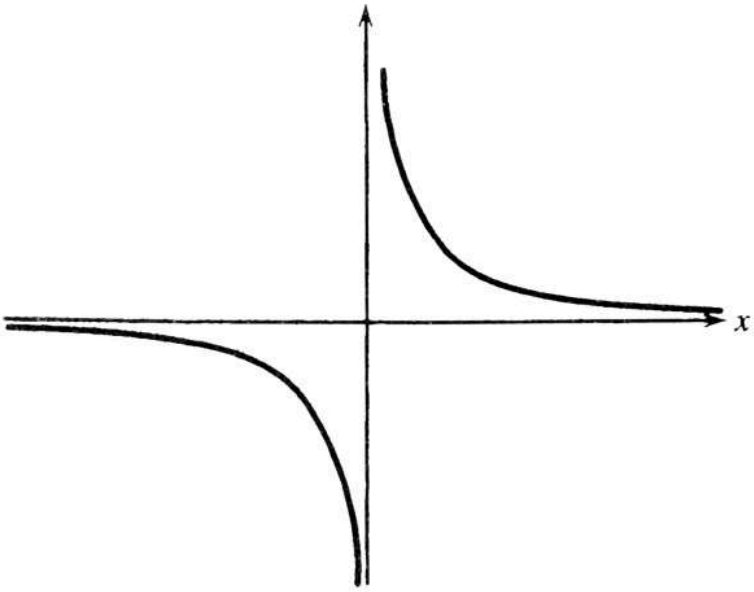

FIGURE 5. Graph of $f(x) = 1/x$ $(x \neq 0)$

is not bounded on a neighborhood of 0, so it cannot have a limit in the sense of the definition of limit given so far. While the symbols $\infty(+\infty)$ and $-\infty$ do not represent real numbers, it is sometimes useful to be able to say that

$$f(x) \quad \text{tends to} \quad \infty \quad as \quad x \longrightarrow 0.$$

This use of $\pm\infty$ will not cause any difficulties, provided we exercise caution and never interpret ∞ and $-\infty$ as being real numbers.

Definition 1.4. Let $f : [a, b] \longrightarrow \mathbb{R}$, and let $c \in [a, b]$.

(a.) We say that f tends to ∞ as $x \longrightarrow c$, and write

$$\lim_{x \longrightarrow c} f(x) = \infty$$

if for every $\alpha \in \mathbb{R}$ there exists $\delta(\alpha) > 0$ such that for all $x \in [a, b]$ with

$$0 < |x - c| < \delta, \text{implies} f(x) > \alpha.$$

(b.) We say that f tends to $-\infty$ as $x \longrightarrow c$, and write

$$\lim_{x \longrightarrow c} f(x) = -\infty$$

if for every $\beta \in \mathbb{R}$ there exists $\delta(\beta) > 0$ such that for all $x \in [a, b]$ with

$$0 < |x - c| < \delta, \text{implies} f(x) < \beta.$$

1.6. **Right-hand Limits and Left-Hand Infinite Limits.**

Definition 1.5. Let $f : [a, b] \longrightarrow \mathbb{R}$, and let $c \in [a, b]$.

(a.) Let $x \in (c, b)$. We say that f **tends to** ∞ **[respectively,** $-\infty$**]** as $x \longrightarrow c+$, and write

$$\lim_{x \longrightarrow c+} f(x) = \infty \text{ [respectively, } \lim_{x \longrightarrow c+} f(x) = -\infty]$$

if for every $\alpha \in \mathbb{R}$ there exists $\delta(\alpha) > 0$ such that for all $x \in [a, b]$ with

$$0 < x - c < \delta, \text{then } \ f(x) > \alpha \ \text{[respectively } \ f(x) < \alpha].$$

(b.) Let $x \in (a, c)$. We say that f **tends to** ∞ **[respectively,** $-\infty$**]** as $x \longrightarrow c-$, and write

$$\lim_{x \longrightarrow c-} f(x) = \infty \text{ [respectively, } \lim_{x \longrightarrow c-} f(x) = -\infty]$$

if for every $\beta \in \mathbb{R}$ there exists $\delta(\beta) > 0$ such that for all $x \in [a, b]$ with

$$0 < c - x < \delta, \text{then } \ f(x) > \beta \ \text{[respectively } \ f(x) < \beta].$$

Example 1.6.

Use the Fig. 4 and Fig. 5 to show the following:
(a.)

$$\lim_{x \longrightarrow 0+} \frac{1}{x} = \infty, \quad x \neq 0.$$

(b.)

$$\lim_{x \longrightarrow 0-} \frac{1}{x} = -\infty, \quad x \neq 0.$$

(c.) Taking cognizance of (a) and (b), can you find

$$\lim_{x \longrightarrow 0} \frac{1}{x}, \quad x \neq 0?$$

(d.)

$$\lim_{x \longrightarrow 0} \frac{1}{x^2} = \infty.$$

TWO USEFUL THEOREMS ON INFINTE LIMITS

Theorem 1.7. *Let $c \in \mathbb{R}$ and let f be defined for $x \in (c, \infty)$ and $f(x) > 0$ for all $x \in (c, \infty)$. Then*

$$\lim_{x \longrightarrow c} f(x) = \infty \quad \textit{if and only if} \quad \lim_{x \longrightarrow c} \frac{1}{f(x)} = 0.$$

Theorem 1.8. *Let $c \in \mathbb{R}$ and let f be defined for $x \in (-\infty, c)$ and $f(x) < 0$ for all $x \in (-\infty, c)$. Then*

$$\lim_{x \longrightarrow c} f(x) = -\infty \quad \textit{if and only if} \quad \lim_{x \longrightarrow c} \frac{1}{f(x)} = 0.$$

CAN YOU APPLY THESE THEOREMS TO CRACK THE QUESTIONS CONTAINED IN THE EXAMPLE 1.6 ABOVE?

1.7. **Limits at Infinity.**

We are going to give the definition of the notion of the limit of a function as $x \longrightarrow \pm\infty$. **(also referred to as as x becomes arbitrary large or small)**.

Definition 1.9. Let $c \in \mathbb{R}$ and let $f : (-\infty, \infty) \longrightarrow \mathbb{R}$,

(a.) We say that $L \in \mathbb{R}$ is a limit of f as $x \longrightarrow \infty$, and write

$$\lim_{x \longrightarrow \infty} f = L \quad \text{or} \quad \lim_{x \longrightarrow \infty} f(x) = L$$

if given any $\epsilon > 0$ there exists $K = k(\epsilon) > c$ such that for any $x > K$, then $|f(x) - L| < \epsilon$.

(b.) We say that $L \in \mathbb{R}$ is a limit of f as $x \longrightarrow -\infty$, and write

$$\lim_{x \longrightarrow -\infty} f = L \quad \text{or} \quad \lim_{x \longrightarrow -\infty} f(x) = L$$

if given any $\epsilon > 0$ there exists $M = M(\epsilon) < c$ such that for any $x < M$, then $|f(x) - L| < \epsilon$.

Example 1.10.

Use the Fig. 4 and Fig. 5 to show the following:

(a.)

$$\lim_{x \longrightarrow \infty} \frac{1}{x} = 0, \quad x \neq 0.$$

(b.)

$$\lim_{x \longrightarrow -\infty} \frac{1}{x} = 0, \quad x \neq 0.$$

(c.)

$$\lim_{x \longrightarrow \infty} \frac{1}{x^2} = 0 = \lim_{x \longrightarrow -\infty} \frac{1}{x^2}.$$

IMPORTANT NOTES ON LIMITS OF POLYNOMIAL FUNCTION AT INFINITY

(a.)

$$\lim_{x\longrightarrow\infty} x^n = \infty, \quad n \in \mathbb{N}.$$

(b.)

$$\lim_{x\longrightarrow\infty} x^n = \begin{cases} \infty & when \;\; n \in \mathbb{N}, n \; even, \\ -\infty & when \;\; n \in \mathbb{N}, n \; odd. \end{cases}$$

(c.) Let $p : \mathbb{R} \longrightarrow \mathbb{R}$ be the polynomial function

$$p(x) := a_n x^n + a_{n-1}x^{n-1} + \cdots a_1 x + a_0.$$

Then

$$\lim_{x\longrightarrow\infty} p(x) = \begin{cases} \infty & for \; a_n > 0, \\ -\infty & for \; a_n < 0. \end{cases}$$

(d.) Let $p : \mathbb{R} \longrightarrow \mathbb{R}$ be the polynomial function

$$p(x) := a_n x^n + a_{n-1}x^{n-1} + \cdots a_1 x + a_0.$$

Then

$$\lim_{x\longrightarrow-\infty} p(x) = \begin{cases} \infty & for \; a_n > 0 \; and \; n \; is \; even, \\ -\infty & for \; a_n > 0 \; and \; n \; is \; odd. \end{cases}$$

SOME USEFUL THEOREMS ON LIMITS AT INFINITY

Theorem 1.11. *Suppose that f and g have limits in $\mathbb{R}$ as $x \longrightarrow \infty$ and that $f(x) \leq g(x)$ for all $x \in (a, \infty)$. Then*

$$\lim_{x\longrightarrow\infty} f(x) \leq \lim_{x\longrightarrow\infty} g(x).$$

Theorem 1.12. *Let f be defined on $(0, \infty)$ to $\mathbb{R}$. Then*

$$\lim_{x\longrightarrow\infty} f(x) = L \quad \text{if and only if} \quad \lim_{x\longrightarrow 0+} f(1/x) = L.$$

Theorem 1.13. *Let f be defined on $(\infty, 0)$ to $\mathbb{R}$. Then*

$$\lim_{x\longrightarrow-\infty} f(x) = L \quad \text{if and only if} \quad \lim_{x\longrightarrow 0-} f(1/x) = L.$$

SOLUTION TO EXAMPLE 1.10

We are going to apply Theorems 1.12 and 1.13.
(a.) By Theorem 1.12,

$$\lim_{x\longrightarrow\infty} \frac{1}{x} = \lim_{x\longrightarrow 0+} x = 0,$$

(b.) By Theorem 1.13,

$$\lim_{x\longrightarrow -\infty} \frac{1}{x} = \lim_{x\longrightarrow 0-} x = 0,$$

(c.) By Theorem 1.12,

$$\lim_{x\longrightarrow \infty} \frac{1}{x^2} = \lim_{x\longrightarrow 0+} x^2 = 0.$$

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LIMITS OF REAL-VALUED FUNCTIONS OF REAL VARIABLE X

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1. PROPERTIES OF LIMITS

In this section, we would attempt to give some properties are very useful methods for finding limits.

1.1. Theorems on Sum, Product and Quotient of Functions.

Theorem 1.1. *$f, g, h : [a, b] \longrightarrow \mathbb{R}$, and let $c \in [a, b]$. Let $k \in \mathbb{R}$. Suppose that*

$$\lim_{x \longrightarrow c} f := \lim_{x \longrightarrow c} f(x) = L \quad \textit{and} \lim_{x \longrightarrow c} g := \lim_{x \longrightarrow c} g(x) = M.$$

Then

(a.)

$$\lim_{x \longrightarrow c} (f \pm g) := \lim_{x \longrightarrow c} (f(x) \pm g(x)) = \lim_{x \longrightarrow c} f(x) \pm \lim_{x \longrightarrow c} g(x) = L \pm M.$$

(b.)

$$\lim_{x \longrightarrow c} (fg) := \lim_{x \longrightarrow c} (f(x)g(x)) = \left(\lim_{x \longrightarrow c} f(x)\right)\left(\lim_{x \longrightarrow c} g(x)\right) = LM.$$

(c.)

$$\lim_{x \longrightarrow c} (bf) := \lim_{x \longrightarrow c} (bf(x)) = b\left(\lim_{x \longrightarrow c} f(x)\right) = bL.$$

(d.)

$$\lim_{x \longrightarrow c} \left(\frac{f}{h}\right) := \lim_{x \longrightarrow c} \left(\frac{f}{h}\right)(x) := \lim_{x \longrightarrow c} \frac{f(x)}{h(x)} := \frac{L}{H},$$

where supposed that $h(x) \neq 0 \;\; \forall \;\; x \in [a, b]$ and

$$\lim_{x \longrightarrow c} h(x) = H \neq 0.$$

Remark 1.2.

(a.) Using inductive argument and the Theorem 2.1(b), we have that

$$\left(\lim_{x \longrightarrow c} f(x)\right)^n = L^n.$$

(b.) From the Theorem 2.1 (a-c), it is easy to see that

$$\lim_{x\longrightarrow c} p(x) := \lim_{x\longrightarrow c} \left(a_n x^n + a_{n-1}x^{n-1} + \cdots a_1 x + a_0\right) \tag{1.1}$$

$$= a_n c^n + a_{n-1}x^{n-1} + \cdots a_1 c + a_0 \tag{1.2}$$

$$= p(c). \tag{1.3}$$

(c.) If $q(c) \neq 0$, applying Theorem 2.1 (d), we have that

$$\lim_{x\longrightarrow c} \frac{p(x)}{q(x)} = \frac{\lim_{x\longrightarrow c} p(x)}{\lim_{x\longrightarrow c} q(x)} = \frac{p(c)}{q(c)},$$

where $p(x) := a_n x^n + a_{n-1}x^{n-1} + \cdots a_1 x + a_0$ and $q(x) := b_n x^n + b_{n-1}x^{n-1} + \cdots b_1 x + b_0$.

1.2. **Limit preserves inequality signs (Ordering).**

Theorem 1.3. *$f : [a, b] \longrightarrow \mathbb{R}$, and let $c \in [a, b]$. If*

$$a \leq f(x) \leq b \ \ \textit{forall} \ \ x \in [a, b], \quad x \neq c,$$

and if

$$\lim_{x\longrightarrow c} f(x) \ \ \textit{exists},$$

then

$$a \leq \lim_{x\longrightarrow c} f(x) \leq b.$$

1.3. **Squeeze Theorem.**

Theorem 1.4. *$f, g, h : [a, b] \longrightarrow \mathbb{R}$, and let $c \in [a, b]$. If*

$$f(x) \leq g(x) \leq h(x) \ \ \textit{forall} \ \ x \in [a, b], \quad x \neq c,$$

and if

$$\lim_{x\longrightarrow c} f(x) = L = \lim_{x\longrightarrow c} h(x),$$

then

$$\lim_{x\longrightarrow c} g(x) = L.$$

1.4. **Limit involving Square roots.**

Theorem 1.5. *$f : [a, b] \longrightarrow \mathbb{R}$, and let $c \in [a, b]$. Suppose that $f(x) \geq 0$ forall $x \in [a, b]$, and let $\sqrt{f}$ be the function defined by*

$$(\sqrt{f})(x) := \sqrt{f(x)} \ \ \textit{forall} \ \ x \in [a, b].$$

If

$$\lim_{x\longrightarrow c} f(x) \ \ \textit{exists}$$

then

$$\lim_{x\longrightarrow c} \sqrt{f} = \sqrt{\lim_{x\longrightarrow c} f}.$$

1.5. **Important Inequalities.**

(a.)

$$x \leq x^{1/2} \leq 1 \text{ holds for } 0 \leq x \leq 1.$$

(b.)

$$-1 \leq \sin x, \cos x \leq 1 \quad \text{holds for all } x \in \mathbb{R}.$$

(c.)

$$1 - 2x^2 \leq \cos x \leq 1 \text{ for all } x \in \mathbb{R}.$$

(d.)

$$x - \frac{1}{6}x^3 \leq \sin x \leq x \text{ for all } x \geq 0.$$

(e.)

$$x \leq \sin x \leq x - \frac{1}{6}x^3 \text{ for all } x \leq 0.$$

(f.)

$$0 < t < e^t \quad \text{for} \quad t > 0.$$

Example 1.6.

Find the following limits

(a.)

$$\lim_{x \longrightarrow 0} \left(\frac{\cos x - 1}{x} \right).$$

(b.)

$$\lim_{x \longrightarrow 0} \left(\frac{\sin x}{x} \right).$$

(c.)

$$\lim_{x \longrightarrow 0} x \sin \left(\frac{1}{x} \right).$$

(d.)

$$\lim_{x \longrightarrow 1} \frac{x^2 + 2}{x^2 - 2}.$$

(e.)

$$\lim_{x \longrightarrow 0} \frac{\sqrt{1 + 2x} - \sqrt{1 + 3x}}{x + 2x^2}.$$

(f.)

$$\lim_{x \longrightarrow 0} \sqrt{\frac{2x + 1}{2x + 3}}.$$

(a.) From the inequality 2.5 (c), we have

$$-2x^2 \leq \cos x - 1 \leq 0 \quad \text{for all} \quad x \in \mathbb{R}.$$

We then that

$$-2x \leq \frac{\cos x - 1}{x} \leq 0 \quad \text{for all} \quad x \neq 0.$$

Observe that $\lim_{x \longrightarrow 0} -2x = 0 = \lim_{x \longrightarrow 0}$. Hence, it follows from squeeze theorem that

$$\lim_{x \longrightarrow 0} \left(\frac{\cos x - 1}{x} \right) = 0.$$

(b.) From inequalities 2.5 (d) and 2.5(e), we have that

$$1 - \frac{1}{6}x^2 \leq \frac{\sin x}{x} \leq 1 \quad \text{for all} \quad x \neq 0.$$

Observe that

$$\lim_{x \longrightarrow 0} \left(1 - \frac{1}{6}x^2 \right) = 1 = \lim_{x \longrightarrow 0} 1.$$

Hence, it follows from squeeze theorem that

$$\lim_{x \longrightarrow 0} \left(\frac{\sin x}{x} \right) = 1.$$

(c.) From 2.5 (b), we have]

$$-1 \leq \sin \left(\frac{1}{x} \right) \leq 1 \qquad \text{holds for all} \quad x \in \mathbb{R}.$$

Clearly, this implies that

$$-x \leq x \sin \left(\frac{1}{x} \right) \leq x \qquad \text{holds for all} \quad x > 0$$

and

$$x \leq x \sin \left(\frac{1}{x} \right) \leq -x \qquad \text{holds for all} \quad x < 0.$$

The last two inequalities above yields

$$-|x| \leq x \sin \left(\frac{1}{x} \right) \leq |x| \qquad \text{for all} \quad x \neq 0.$$

Observe that

$$\lim_{x \longrightarrow 0} -|x| = 0 = \lim_{x \longrightarrow 0} |x|.$$

Hence, it follows from the squeeze theorem that

$$\lim_{x \longrightarrow 0} x \sin \left(\frac{1}{x} \right) = 0.$$

(d.) Observe first that x^2+2, x^2-2 are polynomials and

$$\lim_{x\longrightarrow 1}(x^2-2) = -1 \neq 0.$$

Hence by Remark 2.2 (c), we have that

$$\lim_{x\longrightarrow 1}\frac{x^2+2}{x^2-2} = \frac{\lim_{x\longrightarrow 1}(x^2+2)}{\lim_{x\longrightarrow 1}(x^2-2)} = \frac{3}{-1} = -3.$$

(e.) Suppose $x \neq 0$. By rationalizing and using Theorems 2.1 and 2.5, we find that

$$\begin{aligned}
\lim_{x\longrightarrow 0}\frac{\sqrt{1+2x}-\sqrt{1+3x}}{x+2x^2} &= \lim_{x\longrightarrow 0}\left(\left(\frac{\sqrt{1+2x}-\sqrt{1+3x}}{x+2x^2}\right)\times\left(\frac{\sqrt{1+2x}+\sqrt{1+3x}}{\sqrt{1+2x}+\sqrt{1+3x}}\right)\right).\\
&= \lim_{x\longrightarrow 0}\frac{-x}{(x+2x^2)\left(\sqrt{1+2x}+\sqrt{1+3x}\right)}\\
&= \lim_{x\longrightarrow 0}\frac{-1}{(1+2x)\left(\sqrt{1+2x}+\sqrt{1+3x}\right)}\\
&=\frac{\lim_{x\longrightarrow 0} -1}{\lim_{x\longrightarrow 0}\left((1+2x)\left(\sqrt{1+2x}+\sqrt{1+3x}\right)\right)}\\
&=\frac{\lim_{x\longrightarrow 0} -1}{\lim_{x\longrightarrow 0}(1+2x)\lim_{x\longrightarrow 0}\left(\sqrt{1+2x}+\sqrt{1+3x}\right)}\\
&=\frac{\lim_{x\longrightarrow 0} -1}{\lim_{x\longrightarrow 0}(1+2x)\left(\sqrt{\lim_{x\longrightarrow 0}(1+2x)}+\sqrt{\lim_{x\longrightarrow 0}(1+3x)}\right)}\\
&=\frac{-1}{1(1+1)}\\
&=\frac{-1}{2} = -\frac{1}{2}.
\end{aligned}$$

(f.) $\lim_{x\longrightarrow 0}\sqrt{\frac{2x+1}{2x+3}}$

1.6. **Introduction to L'Hospital's Rules.**

In the part, we will be discussing limits

$$\lim_{x\longrightarrow c}\frac{f(x)}{g(x)}$$

or

$$\lim_{x\longrightarrow \pm\infty}\frac{f(x)}{g(x)},$$

where any of the following holds

$$\lim_{x\longrightarrow c} f(x) = 0 = \lim_{x\longrightarrow c} g(x)$$

$$\lim_{x\longrightarrow c} f(x) = \pm\infty \lim_{x\longrightarrow c} g(x)$$

$$\lim_{x\longrightarrow \pm\infty} f(x) = 0 = \lim_{x\longrightarrow c} g(x)$$

$$\lim_{x\longrightarrow \pm\infty} f(x) = \pm\infty = \lim_{x\longrightarrow c} g(x)$$

In this case, the limit $\lim_{x\longrightarrow c} \frac{f(x)}{g(x)}$ is said to be "indeterminate."Later, We will see that in these cases the limits may not exist or may be any real value, depending on the particular functions f and g.
For example, consider the

$$\lim_{x\longrightarrow 0} \frac{\sin x}{x}.$$

Here, $\lim_{x\longrightarrow 0} \sin x = 0$ and $\lim_{x\longrightarrow 0} x = 0$.
The symbol

$$\frac{0}{0}$$

is used to refer to this kind of situation. However, by our early example 2.6(b), we know that the limit exists, and in fact,

$$\lim_{x\longrightarrow 0} \frac{\sin x}{x} = 1.$$

As such, the indeterminate form $\frac{0}{0}$ can lead to any real number, say L, as a limit. Other indeterminate forms are represented by the symbols

$$1^{\infty}, \quad \frac{\infty}{\infty}, \quad 0\cdot\infty, \quad 0^0, \quad \infty^0, \quad \infty-\infty.$$

These notations correspond to the indicated limiting behavior and juxtaposition of the functions f and g. Our attention will be focused on the indeterminate forms 0=0 and1=1. The other indeterminate cases are usually reduced to the form

$$\frac{\infty}{\infty}, \quad \frac{0}{0},$$

by taking logarithms, exponentials, or algebraic manipulations.

1.7. **First Theorem on the L'Hospital's Rules.**

We will give the Theorem for two-sided limit: the Left-hand limit and Right-Limit is exactly treated the same.

Theorem 1.7. *Let $-\infty \leq a < b \leq \infty$ and let f, g be differentiable on (a, b) such that $g(x) \neq 0$ for all $x \in (a, b)$. Suppose that*

$$\lim_{x\longrightarrow a} f(x) = 0 = \lim_{x\longrightarrow a} g(x).$$

(a.) *If*

$$\lim_{x \longrightarrow a} \frac{f^{'}(x)}{g^{'}(x)} = L \in \mathbb{R},$$

then

$$\lim_{x \longrightarrow a} \frac{f(x)}{g(x)} = L.$$

(b.) *If*

$$\lim_{x \longrightarrow a} \frac{f^{'}(x)}{g^{'}(x)} = L \in \{-\infty, \infty\},$$

then

$$\lim_{x \longrightarrow a} \frac{f(x)}{g(x)} = L.$$

Remark 1.8. (a.) Differentiability of the functions at the point a is not assumed.e point a

Example 1.9.

Evaluate the following limits:

(a.)

$$\lim_{x \longrightarrow 0} \frac{1 - \cos x}{x^2}$$

(b.)

$$\lim_{x \longrightarrow 0} \frac{e^x - 1 - x}{x^2}$$

(c.)

$$\lim_{x \longrightarrow 1} \left(\frac{1}{1 - x} - \frac{1}{\ln x} \right) \qquad x > 1.$$

SOLUTION

(a.) Observe first that

$$\lim_{x \longrightarrow 0} (1 - \cos x) = 0 = \lim_{x \longrightarrow 0} x^2,$$

so that the limit

$$\lim_{x \longrightarrow 0} \frac{1 - \cos x}{x^2}$$

has indeterminate form of

$$\frac{0}{0}.$$

Moreover note that

$$g^{'}(x) = \frac{d}{dx}\left(x^2\right) = 2x \neq 0 \;\; forall \;\; x \neq 0.$$

Hence by Theorem 2.7, we have that

$$\lim_{x \longrightarrow 0} \frac{1-\cos x}{x^2} = \lim_{x \longrightarrow 0} \frac{\frac{d}{dx}(1-\cos x)}{\frac{d}{dx}(x^2)} = \lim_{x \longrightarrow 0} \frac{\sin x}{2x}$$
$$= \lim_{x \longrightarrow 0} \frac{\frac{d}{dx}(\sin x)}{\frac{d}{dx}(2x)}$$
$$= \lim_{x \longrightarrow 0} \frac{\cos x}{2} = \frac{1}{2}$$

Here, we applied the Theorem twice (that is, we applied L'Hospital's rule twice) since

$$\lim_{x \longrightarrow 0} (\sin x) = 0 = \lim_{x \longrightarrow 0} 2x,$$

and $\frac{d}{dx}(2x) = 2 \neq 0, \quad x \neq 0.$

(b.) Observe first that

$$\lim_{x \longrightarrow 0} (e^x - 1 - x) = 0 = \lim_{x \longrightarrow 0} x^2,$$

so that the limit

$$\lim_{x \longrightarrow 0} \frac{e^x - 1 - x}{x^2}$$

has again the indeterminate form of

$$\frac{0}{0}.$$

Similar arguments and twice applications of L'Hospital's rules yields that

$$\lim_{x \longrightarrow 0} \frac{e^x - 1 - x}{x^2} = \lim_{x \longrightarrow 0} \frac{\frac{d}{dx}(e^x - 1 - x)}{\frac{d}{dx}(x^2)} = \lim_{x \longrightarrow 0} \frac{e^x - 1}{2x} = \lim_{x \longrightarrow 0} \frac{\frac{d}{dx}(e^x - 1)}{\frac{d}{dx}(2x)}$$
$$= \lim_{x \longrightarrow 0} \frac{e^x}{2} = \frac{1}{2}.$$

(c.) Observe first that

$$\lim_{x \longrightarrow 1+} \left(\frac{1}{x-1} - \frac{1}{\ln x} \right) = \lim_{x \longrightarrow 1+} \left(\frac{1}{x-1} \right) - \lim_{x \longrightarrow 1+} \left(\frac{1}{\ln x} \right).$$

Now

$$\lim_{x \longrightarrow 1+} (x-1) = 0,$$

so that the

$$\lim_{x \longrightarrow 1+} \frac{1}{x-1} = \infty.$$

Similarly,

$$\lim_{x \longrightarrow 1+} \frac{1}{\ln x} = \infty,$$

since

$$\lim_{x \longrightarrow 1+} \ln x = 0.$$

Thus

$$\lim_{x \longrightarrow 1} \left(\frac{1}{x-1} - \frac{1}{\ln x} \right) = \lim_{x \longrightarrow 1+} \left(\frac{1}{x-1} \right) - \lim_{x \longrightarrow 1+} \left(\frac{1}{\ln x} \right) = \infty - \infty.$$

Sometimes simple algebraic manipulations could help change the indeterminate form $\infty - \infty$ to indeterminate form

$$\frac{0}{0}.$$

For

$$\lim_{x \longrightarrow 1+} \left(\frac{1}{x-1} - \frac{1}{\ln x} \right) = \lim_{x \longrightarrow 1+} \left(\frac{\ln x - x + 1}{(x-1)\ln x} \right) \equiv \frac{0}{0},$$

since

$$\lim_{x \longrightarrow 1+} (\ln x - x + 1) = 0 = \lim_{x \longrightarrow 1+} ((x-1)\ln x)$$

and

$$g^{'}(x) = \frac{d}{dx}((x-1)\ln x) = 1 - \frac{1}{x} + \ln x \neq 0 \quad forall \quad x \neq 1.$$

Hence by Theorem 2.7,

$$\begin{aligned}
\lim_{x \longrightarrow 1+} \left(\tfrac{1}{x-1} - \tfrac{1}{\ln x} \right) &= lim_{x \longrightarrow 1+} \left(\frac{1}{x-1} \right) - \lim_{x \longrightarrow 1+} \left(\frac{1}{\ln x} \right) (\equiv \infty - \infty) \\
&= \lim_{x \longrightarrow 1+} \left(\frac{\ln x - x + 1}{(x-1)\ln x} \right) \text{(By simple algebraic manipulations)} \\
&= \lim_{x \longrightarrow 1+} \frac{\frac{d}{dx}(\ln x - x + 1)}{\frac{d}{dx}((x-1)\ln x)} (Applying\ L'Hospital's\ rules) \\
&= \lim_{x \longrightarrow 1+} \frac{\frac{1}{x} - 1}{1 - \frac{1}{x} + \ln x} \\
&= \lim_{x \longrightarrow 1+} \frac{1-x}{x - 1 + x\ln x} \left(\equiv \frac{0}{0} \right) \\
&= \lim_{x \longrightarrow 1+} \frac{\frac{d}{dx}(1-x)}{\frac{d}{dx}(x - 1 + x\ln x)} (Applying\ L'Hospital's\ rules\ again) \\
&= \lim_{x \longrightarrow 1+} \frac{-1}{2 + \ln x} = -\frac{1}{2}.
\end{aligned}$$

1.8. **Second Theorem on the L'Hospital's Rules.**

We will give the Theorem for two-sided limit: the Left-hand limit and Right-Limit is exactly treated the same.

Theorem 1.10. *Let $-\infty \leq a < b \leq \infty$ and let f, g be differentiable on (a, b) such that $g(x) \neq 0$ for all $x \in (a, b)$. Suppose that*

$$\lim_{x \longrightarrow a} g(x) = \pm\infty.$$

(a.) *If*

$$\lim_{x \longrightarrow a} \frac{f^{'}(x)}{g^{'}(x)} = L \in \mathbb{R},$$

then

$$\lim_{x \longrightarrow a} \frac{f(x)}{g(x)} = L.$$

(b.) *If*

$$\lim_{x \longrightarrow a} \frac{f'(x)}{g'(x)} = L \in \{-\infty, \infty\},$$

then

$$\lim_{x \longrightarrow a} \frac{f(x)}{g(x)} = L.$$

Example 1.11.

Evaluate the following limits:

(a.)

$$\lim_{x \longrightarrow \infty} e^{-x}x^2$$

(b.)

$$\lim_{x \longrightarrow 0+} \frac{\ln(\sin x)}{\ln x}$$

(c.)

$$\lim_{x \longrightarrow \infty} \frac{x - \sin x}{x + \sin x}$$

SOLUTION

(a.)

$$\begin{aligned}
\lim_{x \longrightarrow \infty} e^{-x}x^2 (\equiv 0.\infty) &= \lim_{x \longrightarrow \infty} \frac{x^2}{e^x} \text{ (By simple algebraic manipulations)} \\
&\left(\equiv \frac{\infty}{\infty} \text{indeterminate form}\right) \\
&= \lim_{x \longrightarrow \infty} \frac{2x}{e^x} (Applying\ L'Hospital's\ Rule) \\
&\left(\equiv \frac{\infty}{\infty} \text{indeterminate form}\right) \\
&= \lim_{x \longrightarrow \infty} \frac{2}{e^x} (Applying\ L'Hospital's\ Rule) \\
&= 0 \ (Since \lim_{x \longrightarrow \infty} e^x = \infty.)
\end{aligned}$$

(b.)

$$\lim_{x \longrightarrow 0+} \frac{\ln(\sin x)}{\ln x} \left(\equiv \frac{-\infty}{-\infty} \text{ indeterminate form} \right)$$
$$= \lim_{x \longrightarrow 0+} \frac{\frac{\cos x}{\sin x}}{\frac{1}{x}} (Applying\ L'Hospital's\ Rule)$$
$$= \lim_{x \longrightarrow 0+} \left(\left(\frac{x}{\sin x} \right) \cos x \right)$$
$$= \lim_{x \longrightarrow 0+} \left(\frac{x}{\sin x} \right) \lim_{x \longrightarrow 0+} \cos x$$
$$= 1 \times 1 = 1.$$

(c.)

$$\lim_{x \longrightarrow \infty} \frac{x - \sin x}{x + \sin x} \left(\equiv \frac{\infty}{\infty} \text{ indeterminate form} \right)$$
$$= \lim_{x \longrightarrow \infty} \frac{1 - \cos x}{1 + \cos x} (Applying\ L'Hospital's\ Rule)$$
$$\text{(which does not exist)}$$

However, by rewriting, we have

$$\lim_{x \longrightarrow \infty} \frac{x - \sin x}{x + \sin x} = \lim_{x \longrightarrow \infty} \frac{1 - \frac{\sin x}{x}}{1 + \frac{\sin x}{x}}$$
$$= \frac{\lim_{x \longrightarrow \infty} \left(1 - \frac{\sin x}{x}\right)}{lim_{x \longrightarrow \infty} \left(1 + \frac{\sin x}{x}\right)}$$
$$= \frac{1 - \lim_{x \longrightarrow \infty} \left(\frac{\sin x}{x}\right)}{1 + lim_{x \longrightarrow \infty} \left(\frac{\sin x}{x}\right)}$$
$$= \frac{1 - 0}{1 + 0}$$
$$= 1.$$

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CONTINUITY OF REAL-VALUED FUNCTIONS OF REAL VARIABLE X

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1. CONTINUITY

In this section, we will define what it means to say that a function is continuous at a point c, or on a set.

1.1. Continuity at a point.

Definition 1.1. Let $f : [a, b] \longrightarrow \mathbb{R}$, and let $c \in [a, b]$.We say that f is continuous at c if, given any number $\epsilon > 0$, there exists $\delta(\epsilon) > 0$ such that if x is any point of $[a, b]$ satisfying

$$|x - c| < \delta, \quad \text{then} \quad |f(x) - f(c)| < \epsilon.$$

If f fails to be continuous at c, then we say that f is discontinuous at c.

Remark 1.2. From the definition of limit of function at a point c and the definition of continuity above, follows that

f is said to be continuous at point c if and only if

$$\lim_{x \longrightarrow c} f(x) = f(c).$$

That is the following three conditions must hold for f to be continuous at c:

(1) f must be defined at c ($f(c)$ must be a real number);
(2) the limit of f at c must exist in $\mathbb{R}$ ($\lim_{x \longrightarrow c} f(x)$ must be a real number); and
(3) these two values must be equal.

1.2. Continuity on a Set.

Definition 1.3. Let $f : A \subset \mathbb{R} \longrightarrow \mathbb{R}$. Suppose that $[a, b] \subset A$. f is said to be continuous on the set $[a, b]$ if f is continuous at every point of $[a, b]$.

Example 1.4.

(1) The function

$$f(x) = x^n, \quad n \in \{0, \mathbb{N}\}$$

is continuos on $\mathbb{R}$.

(2) Discuss continuity of the following functions on $\mathbb{R}$.

(a)

$$\frac{1}{x^2 - 3x + 2}$$

(b)

$$\frac{1}{x^2 + 1}$$

(c)

$$f(x) = \begin{cases} 1, & x < -2 \\ \frac{1}{x}, & -2 \leq x \leq 2 \\ -1, & x > 2. \end{cases}$$

(3) Suppose the function

$$f(x) = \begin{cases} 4\cos 2x, & x \leq \frac{-\pi}{2} \\ a\sin x + b, & \frac{-\pi}{2} < x < \frac{\pi}{2} \\ -2\sin x, & x \geq \frac{\pi}{2}. \end{cases}$$

is continuous on $[-\pi, \pi]$. Find the values of a and b.

SOLUTION

(1) Consider the function

$$f(x) = x^n, \quad n \in \{0, \mathbb{N}\}.$$

Firstly, let $n = 0$. Then $f(x) = 1$ is a constant function. We want to prove that $f(x) = 1$ is continuos on $\mathbb{R}$. To do this we begin choosing an arbitrary point c of $\mathbb{R}$. Clearly,

$$\lim_{x \longrightarrow c} f(x) = \lim_{x \longrightarrow c} 1 = 1 = f(c).$$

Therefore, $f(x)$ is continuous at c. But c is arbitrary in $\mathbb{R}$. Hence, $f(x)$ is continuous on $\mathbb{R}$.

Now let $n \in \mathbb{N}$. We want to prove that $f(x) = x^n$ is continuos on $\mathbb{R}$. similarly as before, choose an arbitrary point c of $\mathbb{R}$. By our previous examples under limit,

$$\lim_{x \longrightarrow c} f(x) = \lim_{x \longrightarrow c} x^n = c^n = f(c).$$

Therefore, $f(x)$ is continuous at c. But c is arbitrary in $\mathbb{R}$. Hence, $f(x)$ is continuous on $\mathbb{R}$. That completes the solution to the question.

(2) (a) Observe that the function

$$\frac{1}{x^2 - 3x + 2}$$

is defined on $\mathbb{R}$ except at the points $x = 1$ and $x = 2$, which are the zeroes of its denominator, $x^2 - 3x + 2$. Since

$$\frac{1}{x^2 - 3x + 2}$$

is not defined at $x = 1$ and $x = 2$, it fails to be continuous at $x = 1$ and $x = 2$. Therefore, $x = 1$ and $x = 2$ are the points of discontinuities of

$$\frac{1}{x^2 - 3x + 2}.$$

Thus

$$\frac{1}{x^2 - 3x + 2}$$

is not continuous on $\mathbb{R}$, since it is not continuous at all points of $\mathbb{R}$.

(b) Observe first that the function

$$\frac{1}{x^2 + 1}$$

is defined for all points of $\mathbb{R}$. Moreover, for any arbitrary point c in $\mathbb{R}$, we have

$$\lim_{x \longrightarrow c} f(x) = \lim_{x \longrightarrow c} \frac{1}{x^2 + 1} = \frac{1}{\lim_{x \longrightarrow c}(x^2 + 1)} = \frac{1}{c^2 + 1} = f(c).$$

Consequently, $f(x)$ is continuous at c. By arbitrary argument, $f(x)$ is continuous on $\mathbb{R}$.

(c) Here, the function

$$f(x) = \begin{cases} 1, & x < -2 \\ \frac{1}{x}, & -2 \leq x \leq 2 \\ -1, & x > 2. \end{cases}$$

is piecewise.

NOTE You might want to note in determining whether a piecewise function is continuous on an interval or $\mathbb{R}$, it is sufficient to just test for its continuity at the breaking points.

So for this particular case in hand, we ONLY need to test for continuity at the points $x = -2$ and $x = 2$.

Test at x= -2

Now for $f(x)$ to be continuous at $x = -2$, we must have

$$\lim_{x \longrightarrow -2} f(x) = f(-2).$$

But we have the following

(i)
$$\lim_{x\longrightarrow(-2)-} f(x) = \lim_{x\longrightarrow(-2)-} 1 = 1;$$
(ii)
$$\lim_{x\longrightarrow(-2)+} f(x) = \lim_{x\longrightarrow(-2)+} \frac{1}{x} = -\frac{1}{2};$$
(iii)
$$f(-2) = -\frac{1}{2}.$$
Therefore,
$$\lim_{x\longrightarrow -2} f(x)$$
does not exist, since
$$\lim_{x\longrightarrow(-2)-} f(x) = 1 \neq -\frac{1}{2} = \lim_{x\longrightarrow(-2)+} f(x).$$
As such, function $f(x)$ is not continuous at the point $x = -2$.

Test at x= 2
Now for $f(x)$ to be continous at $x = 2$, we must have
$$\lim_{x\longrightarrow 2} f(x) = f(2).$$
But we have the following
(i)
$$\lim_{x\longrightarrow 2-} f(x) = \lim_{x\longrightarrow 2-} \frac{1}{x} = \frac{1}{2};$$
(ii)
$$\lim_{x\longrightarrow 2+} f(x) = \lim_{x\longrightarrow 2+} -1 = -1;$$
(iii)
$$f(2) = \frac{1}{2}.$$
Therefore,
$$\lim_{x\longrightarrow 2} f(x)$$
does not exist, since
$$\lim_{x\longrightarrow 2+} f(x) = -1 \neq \frac{1}{2} = \lim_{x\longrightarrow 2-} f(x).$$
As such, function $f(x)$ is not continuous at the point $x = 2$. Thus the function
$$f(x) = \begin{cases} 1, & x < -2 \\ \frac{1}{x}, & -2 \leq x \leq 2 \\ -1, & x > 2. \end{cases}$$
is not only discontinuous at the points $x = -2$ and $x = 2$ but also not continuous on $\mathbb{R}$, since it is not continuous at all points of $\mathbb{R}$.

(3) Given that the function

$$f(x) = \begin{cases} 4\cos 2x, & x \leq \frac{-\pi}{2} \\ a\sin x + b, & \frac{-\pi}{2} < x < \frac{\pi}{2} \\ -2\sin x, & x \geq \frac{\pi}{2}. \end{cases}$$

is continuous on $[-\pi, \pi]$, then in partcular it must be continuous at the points $x = \frac{-\pi}{2}$ and $x = \frac{\pi}{2}$ of $[-\pi, \pi]$.

Continuity at point $x = \frac{-\pi}{2}$ implies that

$$\lim_{x \longrightarrow -\frac{\pi}{2}} f(x) = f\left(-\frac{\pi}{2}\right).$$

That is

$$\lim_{x \longrightarrow (-\frac{\pi}{2})+} f(x) = \lim_{x \longrightarrow (-\frac{\pi}{2})-} f(x) = f\left(-\frac{\pi}{2}\right).$$

Evaluating, we find that

(a)

$$\lim_{x \longrightarrow (-\frac{\pi}{2})-} f(x) = \lim_{x \longrightarrow (-\frac{\pi}{2})-} (4\cos 2x) = -4,$$

(b)

$$\lim_{x \longrightarrow (-\frac{\pi}{2})+} f(x) = \lim_{x \longrightarrow (-\frac{\pi}{2})+} (a\sin x + b) = -a + b,$$

(c)

$$f\left(-\frac{\pi}{2}\right) = 4\cos 2\left(-\frac{\pi}{2}\right) = -4.$$

Equating the quantities in items a-c, we obtain

$$-a + b = -4.$$

Similarly, continuity at point $x = \frac{\pi}{2}$ implies that

$$\lim_{x \longrightarrow \frac{\pi}{2}} f(x) = f\left(\frac{\pi}{2}\right).$$

That is

$$\lim_{x \longrightarrow (\frac{\pi}{2})+} f(x) = \lim_{x \longrightarrow (\frac{\pi}{2})-} f(x) = f\left(\frac{\pi}{2}\right).$$

Evaluating, we find that

(i)

$$\lim_{x \longrightarrow (\frac{\pi}{2})-} f(x) = \lim_{x \longrightarrow (\frac{\pi}{2})-} (a\sin x + b) = a + b,$$

(ii)

$$\lim_{x \longrightarrow (\frac{\pi}{2})+} f(x) = \lim_{x \longrightarrow (\frac{\pi}{2})+} (-2\sin x) = -2,$$

(iii)

$$f\left(\frac{\pi}{2}\right) = -2\sin\left(\frac{\pi}{2}\right) = -2.$$

Equating the quantities in items (i-iii), we obtain

$$a + b = -2.$$

Consequently, we have the following two equations involving a and b:

$$-a + b = -4$$

$$a + b = -2.$$

Solving simultaneously, we obtain

$$a = 1, \qquad b = -3.$$

1.3. **Functions with Removable Discontinuity at a Point.**

Definition 1.5. A function $f : [a, b] \longrightarrow \mathbb{R}$ is said to have removable discontinuity at point $c \neq [a, b]$ if the function f is not continuous at a point c (because it is not defined at this point), but the function f has a limit L at the point c.

In such case, we define another function $F : [a, b] \cup \{c\} \longrightarrow \mathbb{R}$ by

$$F(x) = \begin{cases} L & \text{for } x = c, \\ f(x) & \text{for } x \in [a, b]. \end{cases}$$

Clearly, F is continuous at c, since

$$\lim_{x \longrightarrow c} F(x) = L = F(c).$$

Example 1.6. Let the f be defined for all $x \in \mathbb{R}, x \neq 2$, by

$$f(x) = \frac{x^2 + x - 6}{x - 2}.$$

Can f be defined at $x = 2$ in such a way that f is continuous at this point?

SOLUTION

YES if

$$\lim_{x \longrightarrow 2} \frac{x^2 + x - 6}{x - 2} \quad \text{exists}$$

and

NO if

$$\lim_{x \longrightarrow 2} \frac{x^2 + x - 6}{x - 2} \quad \text{does not exist.}$$

Now if $x \neq 2$, then

$$\lim_{x \longrightarrow 2} \frac{x^2 + x - 6}{x - 2} = \lim_{x \longrightarrow 2} \frac{(x + 3)(x - 2)}{x - 2} = \lim_{x \longrightarrow 2} (x + 3) = 5.$$

So the function f has a limit 5 at $x = 2$. So the answer is yes and the required definition is function $F : \mathbb{R} \longrightarrow \mathbb{R}$ defined by

$$F(x) = \begin{cases} 5 & \text{for } \ x = 2, \\ \frac{x^2+x-6}{x-2} & \text{for } \ x \in \{\mathbb{R} \setminus \{2\}\}. \end{cases}$$

Exercises

If $x \in \mathbb{R}$, we define $[[x]]$ to be the greatest integer $n \in \mathbb{Z}$ such that $n \leq x$. (Thus, for example, $[[8.3]] = 8$, $[[\pi]] = 3$, $[[-\pi]] = -4$.) The function $x \longrightarrow [[x]]$ is called the greatest integer function.

Determine the points of continuity of the following functions:

(a)

$$f(x) := [[x]];$$

(b)

$$g(x) := x[[x]];$$

(c)

$$f(x) := [[\sin x]];$$

and

(d)

$$f(x) := [[1/x] \quad x \neq 0.$$

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DEPARTMENT OF MATHEMATICS

MTH 201 LECTURE NOTE

2021/2022 HARMMATAN SEMESTER

TOPIC 1: DIFFERENTIABILITY ON $\Re$

DEFINITION 1: If a function f is defined in an open interval (a, b)(i.e $\quad a < x < b$) containing x_0, then the derivative of f at x_0, written as $f(x_0)$ is given by

$$f'(x_0) = \lim_{h \longrightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h},$$

provided the limit exists. If $f'(x_0)$ exist, then we say that the function is differentiable at x_0 or f has a derivative at x_0.

Right- and Left- Hand Derivatives

The status of the derivative at endpoints of the domain of function f, and in other special circumstances, is clarified by the following definitions.

The right-hand derivative of $f(x)$ at $x = x_0$ is defined as

$$f'_+(x_0) = \lim_{h+ \longrightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h},$$

provided the limit exist or defined.

Similarly, the left-hand derivative of $f(x)$ at $x = x_0$ is defined as

$$f'_-(x_0) = \lim_{h- \longrightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h},$$

if the limit exist. Hence, a function $f(x)$ has a derivative at $x = x_0$ if and only if $f'_+(x_0) = f'_-(x_0)$.

DEFINITION 2: A function f is differentiable on a closed interval $[a, b]$ (i.e $\; a \leq x \leq b$) if it is differentiable in (a, b) and if the following limits:

$$\lim_{h \longrightarrow 0^+} \frac{f(a + h) - f(a)}{h} \quad \text{and} \quad \lim_{h \longrightarrow 0^-} \frac{f(b + h) - f(b)}{h} \quad \text{exist.}$$

Hence, we say that a function is differentiable in a closed interval $[a, b]$ if it is differntiable on (a, b) and its right hand derivative and left hand derivative exist at a and b respectively.

THEOREM 1: If $f(x)$ is differentiable at a domain value x_0, then it is continuous at that value x_0. (NB: The converse of this theorem is not true)

Simply put, Differentiability at a point in a domain implies Continuity in that domain.

Example:

Investigate Differentiability and Continuity of the following functions at the given point:

1. $f(x) = x^{\frac{1}{3}}$ at $x = 0$.
2. $f(x) = |x|$ at $x = 0$.
3. $f(x)=\begin{cases} x^2 \sin\frac{1}{x}, & x \neq 0, \\ 0, & x = 0. \end{cases}$

SOLUTIONS:

Question 1:

We investigate differentiability as

$$f'(x) = \frac{1}{3}x^{\frac{-2}{3}}\bigg|_{x=0} = \infty.$$

Alternatively,

$$f'(0) = \lim_{h\longrightarrow 0}\frac{f(h) - f(0)}{h} = \lim_{h\longrightarrow 0}\frac{x^{\frac{1}{3}} - 0}{h} = \lim_{h\longrightarrow 0} h^{\frac{-2}{3}} = \infty.$$

We check the continuity about $x = 0$ by taking limits as follows:

$$\lim_{x\longrightarrow 0} x^{\frac{1}{3}} = \lim_{x\longrightarrow 0+} x^{\frac{1}{3}} = \lim_{x\longrightarrow 0-} x^{\frac{1}{3}} = 0.$$

Since the limit exists, it implies that $f(x)$ is continous at$x = 0$. Thus, continuity does not imply differentiability.

Question 2:

By definition, the function $f(x)$ may be defined as

$$f(x) = |x| = \begin{cases} -x, & x < 0, \\ 0, & x = 0 \\ x, & x > 0. \end{cases}$$

Hence,

$$\lim_{x\longrightarrow 0}|x|=0, \qquad \lim_{x\longrightarrow 0-}-x=0, \qquad \lim_{x\longrightarrow 0+}x=0.$$

Thus,

$$\lim_{x\longrightarrow 0}|x|=\lim_{x\longrightarrow 0-}-x=\lim_{x\longrightarrow 0+}x=0.$$

Since the limit exists and unique, it implies that $f(x)$ is continous at$x=0$.

We investigate differentiability as The left hand limit

$$\lim_{h\longrightarrow 0-}f'(x)=\lim_{h\longrightarrow 0-}\frac{f(h)-0}{h}=\lim_{h\longrightarrow 0-}\frac{-h-0}{h}=-1$$

, and the right-hand limit

$$\lim_{h\longrightarrow 0+}f'(x)=\lim_{h\longrightarrow 0+}\frac{f(h)-0}{h}=\lim_{h\longrightarrow 0+}\frac{h-0}{h}=1$$

.

Hence, we conclude that the function is continous at $x=0$ but not differentiable at the same point. This the given function is not differentiable for $x=0$ because for $x<0$, the derivative is negative(-1) and for $x>0$, the derivative is positive$(+1)$. So, the left hand derivative and right hand derivative do not match.

Question 3:

We investigate the differentiability as:

$$f'(0)=\lim_{h\longrightarrow 0}\frac{h^2\sin\frac{1}{h}-0}{h}=\lim_{h\longrightarrow 0}h\sin\frac{1}{h}.$$

NB: Using the Squeeze or Sandwitch theorem,

$$\lim_{x\longrightarrow 0}x\sin\frac{1}{x}=0,$$

$$f'(0)=\lim_{h\longrightarrow 0}\frac{h^2\sin\frac{1}{h}-0}{h}=\lim_{h\longrightarrow 0}h\sin\frac{1}{h}=0.$$

We can conclude that the function is continous at $x=0$. The continuity may be confirmed as follows:

$$\lim_{x\longrightarrow 0}x^2\sin\frac{1}{x}=\lim_{x\longrightarrow 0}x\lim_{x\longrightarrow 0}x\sin\frac{1}{x}=\lim_{x\longrightarrow 0}x\times\lim_{x\longrightarrow 0}\frac{\sin\frac{1}{x}}{\frac{1}{x}}=0\times\lim_{x\longrightarrow 0}\frac{\sin\frac{1}{x}}{\frac{1}{x}}=0.$$

This is differentiable at $x=0$ and implies continuity at $x=0$.

EXERCISES

Determine the Differentiability and Continuity of the following functions about the indicated point(s).

$$(i)\ \ f(x) = \begin{cases} x\exp(\frac{-1}{x^2}), & x \neq 0, \\ 0, & x = 0. \end{cases}, \quad (ii) f(x) = x|x|, \qquad x = 0,$$

$$(iii) f(x) = x - |x-1|,\ x = 0, 1, \quad (iv)\ \ |x-3|\ \text{ at }\ x = -1$$

$$(v)\ \ f(x) = \begin{cases} x^3 \sin\frac{1}{x}, & x \neq 0, \\ 2, & x = 0. \end{cases}$$

(vi) Determine the value of p for which the function $f(x) = \begin{cases} x^p \sin\frac{1}{x}, & x \neq 0, \\ 0, & x = 0. \end{cases}$, is differentiable

THEOREM 2: The mean value theorem

If $f(x)$ is continuous in $[a,b]$ and differentiable in (a,b), then there exists a point ξ in (a,b) such that

$$f'(\xi) = \frac{f(b) - f(a)}{b - a}, \qquad a < \xi < b.$$

The above result can be written in various alternative forms; for example, if x and x_0 are in the open interval (a,b), then f

$$f(x) = f(x_0) + f'(\xi)(x - x_0), \ \text{ where }\ x_0 < \xi < x.$$

THEOREM 3: Rolle's theorem

If $f(x)$ is continuous in $[a,b]$ and differentiable in (a,b) and if $f(a) = f(b) = 0$, then there exists a point $\xi\ \in\ (a,b)$ such that $f'(\xi) = 0$.

NB: Rolle's theorem is a special case of the mean value theorem.

EXAMPLE:

1. If the function $f(x) = x(x–2)$ is continous in $[1,2]$ and differentiable in $(1,2)$, determine the value of c for which the Mean value theorem is satisfied.

2. Let $f(x) = 1–(x–1)^{2/3}$, $0 < x < 2$. (a) Is Rolle's theorem satisfied and why? Determine a ξ, where $\xi \in 0 << 2$, which satisfies the mean value theorem?

3. Find the value of c in Rolle's Theorem for the function $f(x) = x^3–3x$ in the closed interval $[-\sqrt{3}, 0]$.

Solution:

1. By the mean value theorem,

$$f'(c) = \tfrac{f(2)-f(1)}{2-1}, \qquad \text{where } \; 1 < c < 2$$
$$\text{where} \quad f'(c) = 2c - 2, \qquad \tfrac{f(2)-f(1)}{2-1} = \tfrac{0-(-1)}{2-1} = 1.$$

Thus,

$$2c - 2 = 1, \quad \Rightarrow \quad c = 3/2.$$

2. For Rolle's theorem to be satisfied,

$$f'(\xi) = 0, \Rightarrow \quad f(0) = f(2) = 0.$$

Thus,

It is easy to verify that $\; f(0) = 0, \;\; f(2) = 0,$ and $f'(\xi) = \dfrac{-2}{3}(x-1)^{\frac{-1}{3}} = 0.$

From the above, there is no finite value of $x = \xi \; \in \; 0 < x < 2$, such that

$$f'(\xi) = \frac{-2}{3}(x-1)^{\frac{-1}{3}} = \frac{-2}{3(x-1)^{\frac{1}{3}}} = 0.$$

3.

$$f'(c) = 3c^2 - 3 = \frac{f(\sqrt{3}) - f(0)}{\sqrt{3} - 0} = \frac{0 - 0}{\sqrt{3} - 0} = 0.$$

Thus,

$$f'(c) = 3c^2 - 3 = 0, \quad \Rightarrow \quad c^2 = 1 \;\text{ and }\; c = \pm 1.$$

FURTHER EXAMPLES

1. Determine the differentiability of the function $f(x) = x^2$ in the closed interval $[0, 1]$.

Soln:

A function $f(x)$ is differentiable in the closed interval $[a, b]$ or $a \leq x \leq b$ if it is differentiable in (a, b) or $a < x < b$ and if

$$\lim_{x \to a+} f(x) \quad \text{and} \quad \lim_{x \to b-} f(x) \quad \text{exist.}$$

Thus, let a point x_0 be such that $0 < x_0 < 1$. Then

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} = \lim_{h \to 0} \frac{(x_0 + h)^2 - x_0^2}{h}$$

$$= \lim_{h \to 0} \frac{2x_0 h + h^2}{h} = \lim_{h \to 0} (2x_0 + h) = 2x_0$$

At the endpoints: $x = a = 0$

$$f'_+(0) = \lim_{h \to 0+} \frac{f(0 + h) - f(0)}{h} = \lim_{h \to 0^+} \frac{h^2 - 0}{h} = \lim_{h \to 0^+} h = 0$$

Similarly at $x = b = 1$

$$f'_-(1) = \lim_{h \to 1-} \frac{f(1 + h) - f(1)}{h} = \lim_{h \to 1-} \frac{(1+h)^2 - f(1)}{h}$$

$$= \lim_{h \to 1-} (2 + h) = 2.$$

The existence of these limits show differentiability.

2. Let $f(x) = \begin{cases} 2x-3, & 0 \le x \le 2 \\ x^2-3 & 2 < x \le 4 \end{cases}$

Discuss (i) continuity and (ii) differentiability.

Soln:

We investigate continuity about the point $x_0 = 2$.

$$\lim_{x \to 2+} f(x) = \lim_{x \to 2+} (x^2-3) = 1$$

$$\lim_{x \to 2-} f(x) = \lim_{x \to 2-} (2x-3) = 1$$

$$\lim_{x \to 2} f(x) = \lim_{x \to 2} (2x-3) = 1.$$

$\therefore$ $f(x)$ is continuous at $x_0 = 2$.

Differentiability on the close interval $[0, 4]$?

$$f_+'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{(2h-3) - (-3)}{h}.$$

$$= \lim_{h \to 0} h = 0$$

$$f_-'(4) = \lim_{h \to 0} \frac{f(4+h) - f(4)}{h}$$

$$= \lim_{h \to 0} \frac{[(4+h)^2 - 3] - 13}{h} = \lim_{h \to 0} (8+h) = 8$$

Since $f_+'(0)$ and $f_-'(4)$ exist, then the $f(x)$ is differentiable.

1. Determine the differentiability of the following functions at the indicated point

(i) $f(x) = |x-3|$ at $x = -1$

(ii) $g(x) = |x-1| + |x+1|$ at $x = 0$

2. Prove that $f(x) = |x+1|$ is continuous at $x = -1$ but not differentiable at $x = -1$

3. For what value of P is

$$f(x) = \begin{cases} x^P \sin 1/x & x \neq 0 \\ 0 & x = 0 \end{cases}$$

differentiable at $x = 0$.

Solution:

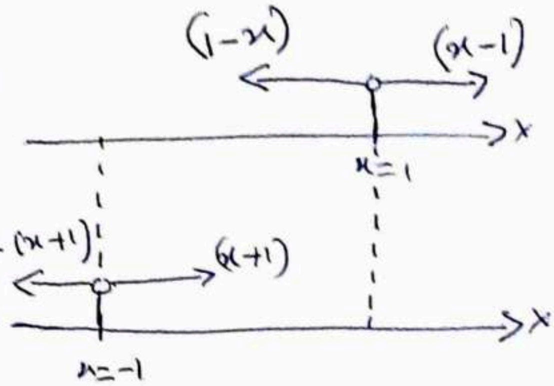
$$f(x) = |x-3| = \begin{cases} -(x-3) & x \leq 3 \\ x-3 & x > 3 \end{cases}$$

$$f(-1) = 3-x\Big|_{x=-1} = 4.$$

$$f'(x) = \begin{cases} -1 & x \leq 3 \\ 1 & x > 3 \end{cases}$$

$$f'(-1) = -1.$$

(ii)



$$g(x) = |x-1| + |x+1| = \begin{cases} 1-x-(x+1), & x<-1 \\ (1-x)+(x+1), & -1<x\leq 1 \\ (x-1)+(x+1), & x>1 \end{cases}$$

$$= \begin{cases} -2x, & x<-1 \\ 2, & -1\leq x\leq 1 \\ 2x, & x>1 \end{cases}$$

$$f'(0) = f'(x)\Big|_{x=0} = 0.$$

NB: Continuity is true at $x=-1,1$.

2.

$$f(x) = \begin{cases} -(x+1) & x\leq -1 \\ (x+1) & x>-1. \end{cases}$$

$$f'(x) = \begin{cases} -1 & x\leq -1 \\ 1 & x> -1 \end{cases}$$

$$f'_{-1^-}(-1) = -1 \quad \text{and} \quad f'_{-1^+}(-1) = 1$$

from the above, the function is not differentiable at $x=-1$ since $f'_{-1^-}(-1) \neq f'_{-1^+}(-1)$.

DEPARTMENT OF MATHEMATICS

OBAFEMI AWOLOWO UNIVERSITY, ILE-IFE.

HARMATTAN SEMESTER 2021/2022 SESSION

MTH 201: MATHEMATICAL METHODS 1. LECTURE NOTE 1

SEQUENCES AND SERIES

1 SEQUENCES

1.1 Convergence of Sequences

We will begin this Lecture with a quick review of Intervals of Real Numbers, $\mathbb{R}$, absolute value and greatest integer functions.

The natural order relation, $' <'$, on $\mathbb{R}$ determines a collection of subsets of $\mathbb{R}$ called intervals which are defined below:

Definition 1.1 *If $a, b \in \mathbb{R}$ satisfies that $a < b$;*

*(i) An **open interval** (a, b) is the subset of $\mathbb{R}$ defined as:*

$$(a, b) := \{x \in \mathbb{R} : a < x < b\}.$$

*(ii) A **closed interval** $[a, b]$ is the subset of $\mathbb{R}$ defined as:*

$$[a, b] := \{x \in \mathbb{R} : a \leq x \leq b\}.$$

*(iii) **Half-open (or Half-closed)intervals** $[a, b)$, $(a, b]$ are the subsets of $\mathbb{R}$ defined as:*

$$[a, b) := \{x \in \mathbb{R} : a \leq x < b\}.$$

$$(a, b] := \{x \in \mathbb{R} : a < x \leq b\}.$$

Unbounded intervals are defined as:

Definition 1.2

$$[a,+\infty) := \{x \in \mathbb{R} : a \leq x < +\infty\}.$$

$$(-\infty, b] := \{x \in \mathbb{R} : -\infty < x \leq b\}.$$

$$(a,+\infty) := \{x \in \mathbb{R} : a < x < +\infty\}.$$

$$(-\infty, b) := \{x \in \mathbb{R} : -\infty < x < b\}.$$

$$(-\infty,+\infty) := \{x \in \mathbb{R} : -\infty < x < +\infty\} = \mathbb{R}.$$

$$[-\infty,+\infty] := \{x \in \mathbb{R} : -\infty \leq x \leq +\infty\} = \mathit{Extended}\ \mathbb{R}.$$

Definition 1.3 *The* ***absolute value function*** $|\,.\,|$ *is a non-negative real-valued function;*

$$|\,.\,| : \mathbb{R} \to [0,+\infty)$$

defined for every $x \in \mathbb{R}$ *as:*

$$|\,x\,| = \begin{cases} x; & if \quad x > 0 \\ 0; & if \quad x = 0 \\ -x; & if \quad x < 0 \end{cases}$$

The following properties hold for absolute value functions.

Proposition 1.1 *(a)* $|\,xy\,| = |\,x\,||\,y\,|$*, for all* $x, y \in \mathbb{R}$

(b) If $\epsilon > 0$ *then* $|\,x\,| \leq \epsilon$ *if and only if* $-\epsilon \leq x \leq \epsilon$ *for all* $x \in \mathbb{R}$.

(c) $-\,|\,x\,| \leq x \leq |\,x\,|$ *for all* $x \in \mathbb{R}$.

Definition 1.4 *The Greatest Integer Function*

The greatest integer function, $[.]$*;*

$$[.] : \mathbb{R} \to \mathbb{Z}$$

is defined as $[x] :=$ *greatest integer less than or equal* x.

For example $[1.7] = 1$, $[0.556] = 0$, $[-0.9] = -1$, $[4] = 4$.

Note that if $x \in \mathbb{Z}$*, then* $[x] = x$.

The greatest integer is also called the step function.

Definition 1.5 *Sequences*

A sequence $\{x\}_{n=1}^{\infty}$ *of real numbers (also real sequence) is a function* f *of the natural numbers* $\mathbb{N}$ *into the real numbers* $\mathbb{R}$,

$$f : \mathbb{N} \to \mathbb{R}; \qquad f(n) = x_n, \ \ \forall \ \ n \in \mathbb{N}.$$

It can also be listed as :

$$x_1, x_2, ..., x_k, ...$$

A Complex sequence is a function $f : \mathbb{N} \to \mathbb{C}$.

Example 1.1 *1.* $\{\sqrt[n]{n}\}_{n=1}^{\infty}$ *2.* $\{(-1)\}_{n=1}^{\infty}$ *, alternating sequence.*
3. $\{5\}$*, constant sequence.* *4.* $1, 2, 3, 5, 8, 13, ...$ *Fibonacci sequence.*
5. $\{\cos\frac{n\pi}{3}\}$*;* $n \in \mathbb{N}$

Definition 1.6 *Convergence of Sequence*

A sequence $\{x_n\}_{n=1}^{\infty}$ of real numbers is said to converge to a real number l if and only if for every $\epsilon > 0$, there exists $n(\epsilon) \in \mathbb{N}$ such that

$$| x_n - l | < \epsilon, \quad \forall n \geq n(\epsilon). \tag{1.1}$$

We say that

$$\lim_{n\to\infty} x_n = l$$

where l is called the limit of the sequence $\{x_n\}_{n=1}^{\infty}$.
We remark that (1.1) is equivalent to

$$x_n \in (l - \epsilon, l + \epsilon), \qquad \forall n \geq n(\epsilon) \tag{1.2}$$

Example 1.2 *Prove that*

$$\lim_{n\to\infty} \frac{1}{n^2 + 1} = 0$$

Solution

Let $\epsilon > 0$ be given. We are to find $n(\epsilon) \in \mathbb{N}$ such that

$$\left| \frac{1}{n^2 + 1} - 0 \right| < \epsilon, \qquad \forall \ n \geq n(\epsilon);$$

But

$$\left| \frac{1}{n^2 + 1} - 0 \right| = \frac{1}{n^2 + 1} < \frac{1}{n}$$

If we choose $n(\epsilon) = [\frac{1}{\epsilon} + 1]$,
then for all $n \geq n(\epsilon) = [\frac{1}{\epsilon} + 1]$,

$$n > \frac{1}{\epsilon}, \qquad \Rightarrow \frac{1}{n} < \epsilon$$

Therefore,

$$\mid \frac{1}{n^2+1} - 0 \mid = \frac{1}{n^2+1} < \frac{1}{n} < \epsilon, \qquad \forall n > n(\epsilon).$$

Hence,

$$\lim_{n\to\infty} \frac{1}{n^2+1} = 0. \qquad \square$$

Example 1.3 *Prove that*

$$\lim_{n\to\infty} \frac{3n+1}{7n-4} = \frac{3}{7}$$

Solution

For any given $\epsilon > 0$, *we want to find* $n(\epsilon) \in \mathbb{N}$ *such that*

$$\mid \frac{3n+1}{7n-4} - \frac{3}{7} \mid < \epsilon, \qquad \forall \;\; n > n(\epsilon)$$

Now,

$$\begin{aligned} \mid \frac{3n+1}{7n-4} - \frac{3}{7} \mid &= \frac{19}{7(7n-4)} \\ &< \frac{19}{7(7n - \frac{7n}{2})} \quad (\textit{where} \;\; n > \frac{8}{7}) \\ &= \frac{19}{7(\frac{7n}{2})} \\ &= \frac{38}{49n} \end{aligned}$$

Therefore, if we choose

$$n(\epsilon) = \max\left\{[\frac{8}{7}+1], [\frac{38}{49\epsilon}+1]\right\}$$

$$\mid \frac{3n+1}{7n-4} - \frac{3}{7} \mid < \epsilon, \qquad \forall \;\; n > n(\epsilon)$$

Example 1.4 *Prove that*

$$If \;\; 0 < a < 1, \;\; then \;\; \lim_{n\to\infty} a^n = 0.$$

Solution

For any given $\epsilon > 0$, *we note that*

$$\begin{aligned} \mid a^n - 0 \mid = a^n &< \epsilon \\ &\Leftrightarrow n \ln a < \ln \epsilon \\ &\Leftrightarrow n > \frac{\ln \epsilon}{\ln a} \end{aligned}$$

Therefore, if we choose

$$n(\epsilon) = [\frac{\ln \epsilon}{\ln a} + 1]$$

For all $n \geq n(\epsilon)$*;* $\mid a^n - 0 \mid < \epsilon$

So, if $0 < a < 1$, $\lim_{n\to\infty} a^n = 0$.

Exercise 1.1 *1. Prove that* (1.1) *is equivalent to* (1.2)

2. Prove the following by finding a suitable $n(\epsilon)$*:*

$$(a) \quad \lim_{n\to\infty} \frac{3n+2}{n+1} = 3$$

$$(b) \quad \lim_{n\to\infty} \frac{(-1)^n}{n} = 0$$

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MTH 201: MATHEMATICAL METHODS 1. LECTURE NOTE 2

SEQUENCES AND SERIES

1 SEQUENCES

1.1 Properties of Limits

The following are some properties of limits of convergence sequences:

1.If a sequence $\{x_n\}$ converges to a limit l, then the limit is unique.

2. Suppose $\{x_n\}$ and $\{y_n\}$ are convergent sequences, and that

$$\lim x_n = x*, \qquad \lim y_n = y*;$$

If

$$x_n \leq y_n, \forall n \in \mathbb{N}, \ \ then \ \ x* \leq y*.$$

3. If $\{x_n\}$ converges to a limit l and $x_n \geq 0$. then $l \geq 0$

4. If $a \leq x_n \leq b$ for all $n \in \mathbb{N}$

A sequence which does not converge to certain real number is said to **diverge**. Examples of divergent sequences are:

$$\{n!\}, \quad \{(-1)^n\}, \quad \{3n\}$$

1.2 Bounded Subsets of Real numbers

Definition 1.1 *A subset S of real numbers is said to be* ***bounded below*** *if there exists $\alpha \in \mathbb{R}$ such that*

$$\alpha \leq x, \qquad \forall x \in S$$

A subset S of real numbers is said to be ***bounded above*** *if there exists $\beta \in \mathbb{R}$ such that*

$$x \leq \beta, \qquad \forall x \in S$$

A subset S is said to be ***bounded*** *if it is bounded above and below.*

Example 1.1 *1. The set* $\mathbb{N} = \{1, 2, 3, 4, ...\}$ *is bounded from below by* 0 *but it is not bounded from above.*

2. The set

$$\{\frac{n}{2n+1} : n \in \mathbb{N}\}$$

is bounded from above by $\frac{1}{2}$ *and from below by* $\frac{1}{3}$.

3. The set $(0, 1)$ *is bounded.*

4. The set $\{(-1)^n\}$ *is bounded below by* -1 *and above by* 1.

Definition 1.2 *Infimum(Greatest Lower Bound)*

If a subset S *of real numbers is bounded below by* α, *then* α *is said to be a* ***lower bound*** *for* S.

For example -1 *is a lower bound for* $(0, 1)$. 0 *is also a lower bound for* $(0, 1)$.

Let S *be a subset of real numbers which is bounded from below.* $\alpha_0 \in \mathbb{R}$ *is said to be the* ***greatest lower bound(glb)*** *or the* ***infimum of*** S***(***$\inf S$***)*** *if*

(i) α_0 *is a lower bound for* S;

(ii) for any other lower bound α' *for* S, $\alpha' \leq \alpha_0$.

For example, if $S = (0, 1)$; *the set of lower bounds is* $(-\infty, 0]$ *and* $Inf S = 0$.

Definition 1.3 *Supremum(Least Upper Bound)*

If a subset S *of real numbers is bounded above by* β, *then* β *is said to be an* ***upper bound*** *for* S.

For example 5 *is an upper bound for* $(0, 1)$. 1 *is also an upper bound for* $(0, 1)$.

Let S *be a subset of real numbers which is bounded from above.* $\beta_0 \in \mathbb{R}$ *is said to be the* ***least upper bound(lub)*** *or the* ***supremum of*** S***(***$\sup S$***)*** *if*

(i) β_0 *is an upper bound for* S;

(ii) for any other upper bound β' *for* S, $\beta_0 \leq \alpha'$.

For example, if $S = (0, 1)$, *the set of upper bounds of* S *is* $[1, +\infty)$ *and the* $\sup S = 1$.

Definition 1.4 *A sequence* $\{x_n\}$ *is said to be bounded if there exists a real number* K *such that* $| x_n | \leq K$, *for all* $n \in \mathbb{N}$

Remark 1.1 *Let* S, A, B *be bounded subsets of real numbers* $\mathbb{R}$ *and let sequences* $\{x_n\}$ *and* $\{y_n\}$ *be bounded. Then*

(i) $\sup\{x_n + y_n\} \leq \sup x_n + \sup y_n$

(ii) $\inf\{x_n + y_n\} \geq \inf x_n + \inf y_n$

(iii) $\sup(-S) = -\inf S$

(iv) $\inf(-S) = -\sup S$

(v) If $A \subseteq B$*, then*

$$\inf B \leq \inf A \leq \sup A \leq \sup B.$$

(vi) $\sup(\frac{1}{S}) = \frac{1}{\inf S}$*, such that* $inf S > 0$.

Example 1.2 *Find the supremum and infimum if they exist,for the following:*

1. $[-1, 5)$ *(2)* $(-1, 5) \cup [7, 100)$

(3). $S = \{\frac{n}{n+1} : n \in \mathbb{N}\}$ *4.* $\{n! : n \in \mathbb{N}\}$.

Solution

1. Let $A = [-1, 5)$ *then,* $\sup A = 5$ *and* $\inf A = -1$.

2. Let $B = (-1, 5) \cup [7, 100)$*, then* $\sup B = 100$*, and* $\inf B = -1$

3. For $S = \{\frac{n}{n+1} : n \in \mathbb{N}\}$*;* $\sup S = 1$ *and* $inf S = \frac{1}{2}$

4. Let $C = \{n! : n \in \mathbb{N}\}$ *then,* $\sup C$ *does not exist and* $\inf S = 1$

Theorem 1.1 *Every convergent sequence is bounded.*

The converse is not necessarily true, that is, not every bounded sequence is convergent, e.g.. $\{(-1)^n\}$.

However, the Bolzano-Weierstrass theorem states that every bounded sequence of real numbers has a convergent subsequence.

Exercise 1.1 *1. Prove the following*

$$(a) \quad \lim_{n\to\infty} (\sqrt{n^2 + 1} - n) = 0$$

$$(b) \quad \lim_{n\to\infty} (\frac{\sqrt{n}}{\sqrt{n+1} + \sqrt{n}}) = \frac{1}{2}$$

2. Find the supremum and infimum of the following sequences:

$$(a)\left\{\frac{n-1}{2n}\right\} \quad (b)\left\{\frac{(-1)^n n}{2n+1}\right\} \quad (c)\left\{\frac{1+(-1)^n}{3}\right\}$$

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DEPARTMENT OF MATHEMATICS

OBAFEMI AWOLOWO UNIVERSITY, ILE-IFE.

HARMATTAN SEMESTER 2021/2022 SESSION

MTH 201: MATHEMATICAL METHODS 1. LECTURE NOTE 3

SEQUENCES AND SERIES

1 SEQUENCES

1.1 Monotone Sequences

Definition 1.1 *A sequence $\{x_n\}$ of real numbers is said to be monotone if for all $n \in \mathbb{N}$, either $x_{n+1} \leq x_n$ or $x_n \leq x_{n+1}$.*
If $x_{n+1} \leq x_n \ \forall n \in \mathbb{N}$*; then $\{x_n\}$ is said to be monotone decreasing.*
If $x_n \leq x_{n+1} \ \forall n \in \mathbb{N}$*; then $\{x_n\}$ is said to be monotone increasing.*

Example 1.1 *1. The sequence $\{a_n\} = \{1 - \frac{1}{n}\}$ is monotone increasing.*
Since

$$a_{n+1} - a_n = \frac{1}{n} - \frac{1}{n+1} \geq 0.$$

2. The sequence $\{n^3\}$ is monotone increasing.
3. $\{(-1)^n\}$ is not monotone.
4. $\{\frac{1}{n^2}\}$ is monotone decreasing.

Remark 1.1 *1. A monotone increasing sequence of real numbers which is bounded above converges.*
2. A monotone decreasing sequence of real numbers which is bounded below converges.

Example 1.2 *1. $\{\frac{1}{\sqrt{n}}\}$ is monotone decreasing, bounded below and its infimum is 0. Therefore it converges to 0.*
2. $\{a^n\}$ is monotone increasing if $a > 1$ and not bounded above. Therefore $\{a^n\}$ diverges for $a > 1$.
3. Let $b_1 = 1$ and $b_{n+1} = \sqrt{2 + b_n}$.
The sequence is monotone increasing, bounded above and its supremum is 2. Therefore the sequence converges to 2.

Exercise 1.1 *1. Establish the convergence and find the limits of the following sequences:*

$$(a)\ \left\{\left(1+\frac{1}{n}\right)^{n+1}\right\} \quad (b)\left\{\left(1-\frac{1}{n}\right)^{n}\right\}$$

2. Determine which of the following sequences are monotone. If it is monotone check whether it is bounded and find its limits.

$$(a)\ \left\{\frac{n^2+1}{n}\right\} \qquad (b)\left\{\frac{2n^2-1}{2n^2+1}\right\} \quad (c)\ \left\{\frac{10^n}{n!}\right\}$$

3. Let $\{a_n\}$ *be defined as* $a_1 = \sqrt{3}$ *and* $a_{n+1} = \sqrt{3a_n}$, $n \geq 1$. *Prove that the sequence* $\{a_n\}$ *converges and find its limit.*

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HARMATTAN SEMESTER 2021/2022 SESSION

MTH 201: MATHEMATICAL METHODS 1. LECTURE NOTE 4

SEQUENCES AND SERIES

1 SERIES

1.1 Definition and Examples

Definition 1.1 *A series $\sum_{i=1}^{\infty} x_i$ of real numbers is the sum of sequence,*

$$x_1, x_2, ..., x_n, ...$$

Given a series $\sum_{i=1}^{\infty} x_i$, the nth partial sum of the series is

$$s_n = \sum_{i=1}^{n} x_i$$

A series $\sum_{i=1}^{\infty} x_i$ is said to converge to a limit l if its sequence of partial sums $\{s_n\}$ converges to l. Otherwise the series is said to diverge.

Example 1.1 *1. The series $\sum_{i=1}^{\infty}(\frac{2}{3})^i$ converges to 2.*

Solution

The sequence of partial sums is :

$$s_1 = \frac{2}{3}$$
$$s_2 = \frac{2}{3} + (\frac{2}{3})^2$$
$$s_3 = \frac{2}{3} + (\frac{2}{3})^2 + (\frac{2}{3})^3$$

.

.

.

$$\begin{aligned} s_n &= \frac{2}{3} + (\frac{2}{3})^2 + (\frac{2}{3})^3 + ... + (\frac{2}{3})^n \\ &= \frac{2(1-(\frac{2}{3})^n)}{3(\frac{1}{3})} \\ &= 2(1-(\frac{2}{3})^n) \end{aligned}$$

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} 2(1-(\frac{2}{3})^n) = 2$$

2. Determine whether or not the series

$$\frac{1}{2.5}+\frac{1}{5.8}+\frac{1}{8.11}+... \qquad \textit{converges}$$

Solution

The nth term is $\frac{1}{(3n-1)(3n+2)}$.

$$\begin{aligned} s_n &= \sum_{r=1}^{n} \frac{1}{(3r-1)(3r+2)} \\ &= \frac{1}{3}\sum_{r=1}^{n}\left(\frac{1}{3r-1}-\frac{1}{3r+2}\right) \\ &= \frac{1}{2}-\frac{1}{5}+\frac{1}{5}-\frac{1}{8}+...+\frac{1}{3n-1}-\frac{1}{3n+2} \\ &= \frac{1}{2}-\frac{1}{3n+2} \end{aligned}$$

Therefore,

$$s_n = \frac{1}{3}\left(\frac{1}{2}-\frac{1}{3n+2}\right)$$

$$\lim_{n\to\infty} s_n = \frac{1}{6}$$

The sequence of partial sums converges to $\frac{1}{6}$, *therefore the series converges to* $\frac{1}{6}$.

3. Prove that the series

$$\sum_{n=1}^{\infty}\frac{n}{(n+1)!} \quad \textit{converges to } 1$$

Solution

The sequence of partial sums $\{s_n\}$ *for the series is:*

$$\begin{aligned} s_n &= \sum_{r=1}^{n}\frac{r}{(r+1)!} \\ &= \sum_{r=1}^{n}\left(\frac{1}{r!}-\frac{1}{(r+1)!}\right) \\ &= 1-\frac{1}{(n+1)!} \end{aligned}$$

Therefore,

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty}\left(1-\frac{1}{(n+1)!}\right) = 1$$

Remark 1.1 *1. If a series* $\sum_{n=1}^{\infty} x_n$ *is convergent, then* $\lim_{n\to\infty} x_n = 0$.

The result is necessary but not sufficient condition for convergence of a series. For example, the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ *diverges but* $\lim_{n\to\infty} \frac{1}{n} = 0$.

2. If $\lim_{n\to\infty} x_n \neq 0$ *then the series* $\sum_{n=1}^{\infty} x_n$ *is divergent.*

3. If $\sum_{n=1}^{\infty} | x_n |$ *converges, then the series is said to converge absolutely.*

4. If $\sum_{n=1}^{\infty} x_n$ *converges and* $\sum_{n=1}^{\infty} | x_n |$ *diverges, then* $\sum_{n=1}^{\infty} x_n$ *is said to converge conditionally.*

5. If a series $\sum_{n=1}^{\infty} x_n$ *is absolutely convergent, then it converges and*

$$| \sum_{n=1}^{\infty} x_n | \leq \sum_{n=1}^{\infty} | x_n |$$

1.2 Tests of Convergence

1. **Comparison test**

Let $\sum_{n=1}^{\infty} x_n$ and $\sum_{n=1}^{\infty} y_n$ be two series with non-negative terms.

Suppose for some positive integer n_0;

$$x_n \leq y_n, \forall n > n_0; \quad \text{Then}$$

(i)if the series $\sum_{n=1}^{\infty} y_n$ converges, then the series $\sum_{n=1}^{\infty} x_n$ converges;

(ii) if the series $\sum_{n=1}^{\infty} x_n$ diverges then the series $\sum_{n=1}^{\infty} y_n$ diverges.

Example 1.2 *1. Consider the series*

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} \quad \textit{and} \quad \sum_{n=1}^{\infty} \frac{1}{n^2}$$

The two series have nonnegative terms and

$$\frac{1}{n(n+1)} < \frac{1}{n^2}, \qquad \forall n$$

Since the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ *converges and*

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} \ll \sum_{n=1}^{\infty} \frac{1}{n^2},$$

then the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ *converges.*

2.Consider the series

$$\sum_{n=1}^{\infty} \frac{1}{n} \quad \textit{and} \quad \sum_{n=1}^{\infty} \frac{1}{\ln n}$$

The two series have nonnegative terms and

$$\frac{1}{n} < \frac{1}{\ln n}, \qquad \forall n$$

Since the series $\sum_{n=1}^{\infty} \frac{1}{n}$ *diverges and*

$$\sum_{n=1}^{\infty} \frac{1}{n} \ll \sum_{n=1}^{\infty} \frac{1}{\ln n},$$

then the series $\sum_{n=1}^{\infty} \frac{1}{\ln n}$ *diverges.*

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MTH 201: MATHEMATICAL METHODS 1. LECTURE NOTE 5

SEQUENCES AND SERIES

1 SERIES

1.1 Tests of Convergence

2. **Integral test**

If (i) $x_n \geq 0,\ \ x_n \geq x_{n+1}, \forall n \geq b,\ b \in \mathbb{R}$

(iii) $\lim x_n = 0$. Then

$$\int_b^\infty f(t)dt \text{ converges} \qquad \Leftrightarrow \sum_b^\infty x_n \quad \textit{converges}, \text{ where } x_n = f(t).$$

Example 1.1 *(P-series)*

Show that the series

$$\sum_{n=1}^\infty \frac{1}{n^p} \quad \textit{converges if } p > 1 \textit{ and diverges if } p \leq 1.$$

Solution

Let $f(t) = \frac{1}{t^p}$, *so that for* $t \geq 1$, *the function* f *is nonnegative, monotone decreasing and integrable. Setting*

$$x_n = f(n) = \frac{1}{n^p}, \quad \forall n \in \mathbb{N},$$

then by the integral test,

$$\sum_{n=1}^\infty x_n \ \textit{and} \ \int_1^\infty f(t)dt$$

converge or diverge together. But,

$$\int_1^N f(t)dt = \int_1^N \frac{1}{t^p}dt = \begin{cases} \frac{N^{1-p}-1}{1-p}; & \textit{if} \quad p \neq 1 \\ \log N; & \textit{if} \quad p = 1 \end{cases}$$

Therefore,

$$\int_1^\infty f(t)dt = \lim_{N\to\infty} \int_1^N f(t)dt = \begin{cases} \frac{-1}{1-p}; & \textit{if} \quad p > 1 \\ \infty; & \textit{if} \quad 0 < p \leq 1. \end{cases}$$

Thus

$$\int_1^\infty f(t)dt$$

converges if $p > 1$ *and diverges if* $0 < p \leq 1$.

But when $p < 0$, *the given series diverges, because, the nth term* $\frac{1}{n^p}$ *does not tend to zero as* $n \to \infty$.

Hence the series

$$\sum_{n=1}^\infty \frac{1}{n^p},$$

converges when $p > 1$ *and diverges when* $p \leq 1$

3. **The Ratio test**

Let $\sum_{n=1}^\infty u_n$ be a series of real numbers.

Suppose $\lim_{n\to\infty} \mid \frac{u_{n+1}}{u_n} \mid = L$. Then the series $\sum_{n=1}^\infty u_n$

(i) converges absolutely if $L < 1$;

(ii) diverges if $L > 1$;

(iii) when $L = 1$, the ratio test fails, since the series may or may not converge.

Example 1.2 *1. The series*

$$\sum_{n=1}^\infty \frac{1}{n!} \quad \textit{converge.}$$

By ratio test,

$$\lim_{n\to\infty} \mid \frac{u_{n+1}}{u_n} \mid = \lim_{n\to\infty} \frac{1}{n+1} = 0.$$

2. The series

$$\sum_{n=1}^\infty \frac{n!}{2^n} \qquad \textit{diverges}$$

By ratio test,

$$\lim_{n\to\infty} \mid \frac{u_{n+1}}{u_n} \mid = \lim_{n\to\infty} \frac{n+1}{2} = \infty$$

4. **Raabe's test**

Let the series $\sum_{n=1}^\infty$ be such that

$$\lim_{n\to\infty} n\left(1 - \mid \frac{u_{n+1}}{u_n} \mid\right) = L$$

Then the series

(i) converges absolutely if $L > 1$;

(ii) diverges or converges conditionally if $L < 1$.

The test fails if $L = 1$.

Example 1.3 *Consider the series*

$$\sum_{n=1}^{\infty} \frac{1.4.7....(3n-2)}{3.6.9....3n}.$$

The ratio test fails, but by Raabe's test

$$\begin{aligned}\lim_{n\to\infty} n\Big(1 - \frac{u_{n+1}}{u_n}\Big) &= \lim_{n\to\infty} n\Big(1 - \frac{3n+1}{3n+3}\Big) \\ &= \lim_{n\to\infty} \frac{2n}{3n+1} \\ &= \frac{2}{3}.\end{aligned}$$

Hence the series diverges.

5. **Gauss'test**

If the series $\sum_{n=1}^{\infty} u_n$ is such that

$$\mid \frac{u_{n+1}}{u_n} \mid = 1 - \frac{\alpha}{+} \frac{\beta}{n^2} \mid$$

then the series

converges absolutely, if $\alpha \leq 1$

diverges or converges conditionally, if $\alpha\rangle 1$

The **nth root test**

Given that the series $\sum_{n=1}^{\infty}$,

If

$$\lim_{n\to\infty} \sqrt[n]{\mid u_n \mid} = \alpha,$$

then the series $\sum_{n=1}^{\infty} u_n$

(i) converges absolutely if $\alpha < 1$

(ii) diverges if $\alpha > 1$

The test fails if $\alpha = 1$

Example 1.4 *Consider the series*

$$\sum_{n=1}^{\infty} \frac{1}{n^n}$$

The series is convergent since,

$$\lim_{n\to\infty} \sqrt[n]{\mid u_n \mid} = \lim_{n\to\infty} \frac{1}{n} = 0.$$

6.**Weiersrass' M test of uniform convergence**

The series of functions $\sum f_n$ converges uniformly on $S \subseteq \mathbb{R}$ if

$$\| f_n \|_S \leq M_n, \qquad n \geq k,$$

for some fixed $k \in \mathbb{N}$, where $\sum M_n < \infty$.

Example 1.5 *1. Show that series $\sum \frac{\sin nx}{n^2}$ converge uniformly on $(-\infty, \infty)$.*

Solution

Take $M_n = \frac{1}{n^2}$ and note that $\sum \frac{1}{n^2} < \infty$

$$\begin{aligned} \| f_n \|_{(-\infty,\infty)} &= \| \frac{\sin nx}{n^2} \| \\ &\leq \frac{1}{n^2} \\ &= M_n < \infty. \end{aligned}$$

2. Show that the series of functions

$$\sum f_n(x) = \sum \left(\frac{x}{1+x} \right)^n$$

converges uniformly on any set S such that

$$| \frac{x}{1+x} | \leq r < 1; \qquad x \in S \tag{1.1}$$

Solution

Now, since S is such that

$$\| f_n \|_S \leq r^n.$$

Weierstrass'test applies with

$$\sum M_n = \sum r^n < \infty.$$

Since (1.1) *is equivalent to*

$$\frac{-r}{1+r} \leq x \leq \frac{r}{1-r}, \qquad x \in S.$$

This means that the series converges uniformly on any compact subset $(\frac{-1}{2}, \infty)$; the series does converge uniformly on $(\frac{-1}{2}, b)$ with $b < \infty$ or on $S = [a, \infty)$ with $a > \frac{-1}{2}$, because in these cases, $\| f_n \|_S = 1$, for all n.

7. **Convergence of Power Series**

A Power series is an infinite series of the form

$$\sum_{n=0}^{\infty} a_n(x - x_0)^n,$$

where x_0 is the centre, a_n are coefficients.

The radius of convergence R for a power series $\sum_{n=0}^{\infty} a_n(x - x_0)^n$ is :

$$R = \lim_{n\to\infty} \mid \frac{a_n}{a_{n+1}} \mid,$$

provided the limit exists. The interval of convergence is $\mid x - x_0 \mid < R$ and the series diverges in the interval $\mid x - x_0 \mid > R$.

A Power series converges uniformly and absolutely in any interval which lies entirely within the interval of convergence.

Example 1.6 *For what values of x do the following series converge?*

$$1.\ \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad 2. \sum_{n=1}^{\infty} \frac{n(x-1)^n}{2^n(3n-1)}$$

Solution

1. $u_n = \frac{x^n}{n!}$. *Therefore,*

$$\lim_{n\to\infty} \mid \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \mid \frac{x^{n+1} n!}{x^n(n+1)!} \mid$$
$$= \lim_{n\to\infty} \mid \frac{x}{(n+1)} \mid$$

It implies $\frac{1}{R} = 0$, *therefore, the radius of convergence* $R = \infty$. *That is, the interval of convergence is:*

$$\{x \in \mathbb{R} : \mid x \mid < \infty\}$$

So, the series converges for all real number x. That is, it converges absolutely uniformly in every bounded set.

2. $u_n = \frac{n(x-1)^n}{2^n(3n-1)}$. *Therefore,*

$$\lim_{n\to\infty} \mid \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \mid \frac{(n+1)2^n(3n+1)(x-1)^{n+1}}{n2^{n+1}(3n+2)(x-1)^n} \mid$$
$$= \lim_{n\to\infty} \mid \frac{(n+1)(3n+1)(x-1)}{2n(3n+2)} \mid$$
$$= \mid \frac{x-1}{2} \mid$$

Therefore, the series converges for $|x-1|<2$ *and diverges for* $|x-1|>2$.
The test fails if $|x-1|=2$ *i.e.* $x=-1$ *or* $x=3$. *However, we can test further with other convergence test to establish the absolute convergence or otherwise for the series when* $|x-1|=2$.

Exercise 1.1 *For what values of* x *do the following series converge?*

1. $\displaystyle\sum_{n=0}^{\infty} n!x^n$
2. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n-1}\left(\frac{x+2}{x-1}\right)^n$
3. $\displaystyle\sum_{n=1}^{\infty} \frac{x^{n-1}}{n3^n}$

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Partial Derivatives.

Students are familiar with the concept of differentiability of functions of one real independent variable, $y = f(x)$. The methods of differentiating such functions were discussed in MTH102 (Elementary Mathematics II).

We now extend these ideas to functions of two or more real independent variables $f(x, y, u, v, \ldots)$.

Consider the function f of two variables x and y defined by

$$f(x,y) = x^2 + 2xy - 3y^2.$$

We can determine the value of $f(x,y)$ for every pair of numbers (x,y).

For example, if $(x,y) = (1,1)$ then

$$f(x,y) = 1^2 + 2(1)(1) - 3(1)^2$$
$$= 0.$$

If $f(x,y)$ has just one real value for any pair (a,b) within its region of definition $\mathcal{R}$, then $f(x,y)$ is said to be a single-valued function of

Question:......

x and y within $\mathcal{R}$. If two or more values of the function are obtained for a given (a,b), we call the function two-valued or many-valued.

For instance, the function
$$f(x,y) = x^2 + 2xy - 3y^2$$
is single-valued over the region $\mathcal{R}$ given by $-\infty < x < \infty$, $-\infty < y < \infty$, whereas the function
$$f(x,y) = \sqrt{(a^2 - x^2 - y^2)}$$

is two-valued over the region $\mathcal{R}$ given by $x^2 + y^2 < a^2$, and single-valued on the boundary of the circle $x^2 + y^2 = a^2$.

* Exercise: What happens when
$$x^2 + y^2 > a^2 \ ?$$

Limits

Let $f(x,y)$ be defined in a deleted δ-neighbourhood of (a,b). A number l is said to be the limit of $f(x,y)$ as x approaches a and y approaches b [i.e., as (x,y) approaches (a,b)] if for any positive number ε we

Question.......... Do not write in this margin

can find some positive number δ (depending on ε and (a,b) in general) such that

$$|f(x,y) - l| < \varepsilon \text{ whenever}$$
$$0 < |x-a| < \delta \text{ and } 0 < |y-b| < \delta.$$

Note: The word "deleted" in the definition means $f(x,y)$ may not be defined at (a,b).

When the limit exists, we write

$$\lim_{(x,y)\to(a,b)} f(x,y) = l \quad \text{or} \quad \lim_{\substack{x\to a\\ y\to b}} f(x,y) = l$$

The concept of one-sided limits for functions of one variable can easily be extended to functions of several variables.

For example, $\lim\limits_{\substack{x\to 0^+\\ y\to 1}} \tan^{-1}(y/x) = \pi/2$,

whereas, $\lim\limits_{\substack{x\to 0^-\\ y\to 1}} \tan^{-1}(y/x) = -\pi/2$.

Thus, $\lim\limits_{\substack{x\to 0\\ y\to 1}} \tan^{-1}(y/x)$ does not exist.

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Question

Continuity

Let $f(x, y)$ be defined in a δ-neighbourhood of (a, b). We say that $f(x, y)$ is continuous at (a, b) if for any positive number ε we can find some positive number δ (depending on ε and (a, b) in general) such that

$$|f(x,y) - f(a,b)| < \varepsilon \quad \text{whenever}$$
$$|x-a| < \delta \text{ and } |y-b| < \delta$$

Note that three conditions must be satisfied for $f(x, y)$ to be continuous at (a, b):

1. The limit exists as $(x, y) \to (a, b)$
2. $f(a, b)$ exists, i.e., $f(x, y)$ is defined at (a, b).
3. $\lim\limits_{\substack{x \to a \\ y \to b}} f(x, y) = f(a, b)$.

Remark: The two conditions $|x-a| < \delta$ and $|y-b| < \delta$ can be replaced with $(x-a)^2 + (y-b)^2 < \delta^2$

Example: If $f(x,y) = \begin{cases} 2xy, & (x,y) \neq (1,2) \\ 0, & (x,y) = (1,2), \end{cases}$

Question.. Do not write in this margin

then $\lim_{\substack{x \to 1 \\ y \to 2}} f(x,y) = 2(1)(2) = 4$.

Meanwhile, $f(1,2) = 0$. Hence, $f(x,y)$ is not continuous at $(1,2)$.
If we redefine the function so that $f(x,y) = 4$ for $(x,y) = (1,2)$, then it becomes continuous at $(1,2)$.

Exercise

1. Determine whether $f(x,y) = \begin{cases} x^2+2y; & (x,y) \neq (1,2) \\ 0; & (x,y) = (1,2) \end{cases}$
(a) has a limit as $x \to 1, y \to 2$;
(b) is continuous at $(1,2)$.

2. Investigate the continuity of
$$f(x,y) = \begin{cases} \dfrac{x^2-y^2}{x^2+y^2}; & (x,y) \neq (0,0) \\ 0; & (x,y) = (0,0) \end{cases}$$
at $(0,0)$.

First Partial Derivatives

The partial derivative of the function $f(x, y)$ with respect to x is defined as

$$\frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \left\{ \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x} \right\}$$

Similarly the partial derivative of $f(x, y)$ with respect to y is defined as

$$\frac{\partial f}{\partial y} = \lim_{\Delta y \to 0} \left\{ \frac{f(x, y+\Delta x) - f(x,y)}{\Delta y} \right\}$$

In other words, the partial derivative of a function f of several variables with respect to one of the independent variables, say x, is simply the ordinary derivative of f with respect to x, keeping all other variables constant.

For example, (1) if $f(x, y) = x^2 - 2y^2$, then

$$\frac{\partial f}{\partial x} = 2x \quad \text{and} \quad \frac{\partial f}{\partial y} = -4y.$$

Using the definition (i.e., from first principles) we obtain

Question

$$\frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \left\{ \frac{[(x+\Delta x)^2 - 2y^2] - (x^2 - 2y^2)}{\Delta x} \right\}$$

$$= \lim_{\Delta x \to 0} \left(\frac{2x\Delta x + (\Delta x)^2}{\Delta x} \right) = 2x$$

Similarly,

$$\frac{\partial f}{\partial y} = \lim_{\Delta y \to 0} \left\{ \frac{[x^2 - 2(y+\Delta y)^2] - (x^2 - 2y^2)}{\Delta y} \right\}$$

$$= \lim_{\Delta y \to 0} \left(\frac{-4y\Delta y - 2(\Delta y)^2}{\Delta y} \right) = -4y.$$

(2) If $f(x,y,z) = e^{2z}\cos(xy)$, then

$$\frac{\partial f}{\partial x} = -ye^{2z}\sin(xy);$$

$$\frac{\partial f}{\partial y} = -xe^{2z}\sin(xy);$$

$$\frac{\partial f}{\partial z} = 2e^{2z}\cos(xy).$$

Notation

The partial derivative $\frac{\partial f}{\partial x}$ is

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Question.........

often written as f_x. If $f(x,y)$ is a function of independent variables x and y. We sometimes write

$$\left(\frac{\partial f}{\partial x}\right)_y$$

to denote $\frac{\partial f}{\partial x}$. The subscript y emphasizes that y is kept constant.

e.g., If $f(x,y,z) = 2x^2y - yz$,

then $f_x = \frac{\partial f}{\partial x} = \left(\frac{\partial f}{\partial x}\right)_{y,z} = 4xy$;

$$f_y = \frac{\partial f}{\partial y} = \left(\frac{\partial f}{\partial y}\right)_{x,z} = 2x^2 - z$$

and

$$f_z = \frac{\partial f}{\partial z} = \left(\frac{\partial f}{\partial z}\right)_{x,y} = -y$$

*Exercise: What is the difference between $\frac{\partial y}{\partial x}$ and $\frac{\delta y}{\delta x}$?

(2) If $f(x,y) = 3xy^2 + \sin(xy)$, find f_x and f_y.

Question..............

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Composite Functions

Let $z = f(x,y)$, where $x = g(u,v)$ and $y = h(u,v)$. Then

$$\frac{\partial z}{\partial u} = \frac{\partial x}{\partial u} \cdot \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial u},$$

and

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial v}.$$

This can be generalized to the case where z is a function of n variables: $z = F(x_1, x_2, \ldots, x_n)$, where

$x_i = f_i(u_1, u_2, \ldots, u_p)$; $1 \le i \le n$, and both p and n are some positive integers. Then

$$\frac{\partial z}{\partial u_k} = \frac{\partial z}{\partial x_1} \cdot \frac{\partial x_1}{\partial u_k} + \frac{\partial z}{\partial x_2} \cdot \frac{\partial x_2}{\partial u_k} + \cdots + \frac{\partial z}{\partial x_n} \cdot \frac{\partial x_n}{\partial u_k}$$

$$k = 1, 2, \cdots, p.$$

If, in particular, $x_1, x_2, \ldots, x_n$ depend on only one variable u (i.e., if $x_i = f_i(u)$), then

$$\frac{dz}{du} = \frac{\partial z}{\partial x_1} \cdot \frac{dx_1}{du} + \frac{\partial z}{\partial x_2} \cdot \frac{dx_2}{du} + \cdots + \frac{\partial z}{\partial x_n} \cdot \frac{dx_n}{du}.$$

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Example : $f(x,y) = \tan^{-1}(y/x)$.

Putting $u = y/x$, we have $f(u) = \tan^{-1}u$.

So,

$$\frac{\partial f}{\partial x} = \frac{d(\tan^{-1}u)}{du} \cdot \frac{\partial u}{\partial x}$$

$$= \frac{x^2}{x^2+y^2} \cdot \frac{-y}{x^2}$$

$$= \frac{-y}{x^2+y^2}.$$

and

$$\frac{\partial f}{\partial y} = \frac{d(\tan^{-1}u)}{du} \cdot \frac{\partial u}{\partial y}$$

$$= \frac{x^2}{x^2+y^2} \cdot \frac{1}{x}$$

$$= \frac{x}{x^2+y^2}.$$

Exercise

(1) Given that $u = x^2y + \frac{1}{y}$ and $y = \log_e x$, find $\frac{du}{dx}$.

(2) If $f(u) = \sin u$ and $u = \sqrt{(x^2+y^2)}$, find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

Question..........

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Higher Partial Derivatives.

If $f(x,y)$ has partial derivatives at each point (x,y) in a region, then f_x and f_y are functions of x and y, which may also have partial derivatives. These second derivatives are denoted by:

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = f_{xx} \; ;$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = f_{yy} \; ;$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = f_{yx} \; ;$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = f_{xy} \, .$$

Example

(1) Let $f(x,y) = \tan^{-1}(y/x)$. Since

$$f_x = \frac{-y}{x^2+y^2} \quad \text{and} \quad f_y = \frac{x}{x^2+y^2}$$

then, $f_{xx} = \frac{\partial}{\partial x}\left(\frac{-y}{x^2+y^2}\right) = \frac{2xy}{(x^2+y^2)^2}$,

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$$f_{yy} = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{x}{x^2+y^2}\right) = \frac{-2xy}{(x^2+y^2)^2},$$

$$f_{xy} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right) = \frac{y^2-x^2}{(x^2+y^2)^2},$$

$$f_{yx} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) = \frac{y^2-x^2}{(x^2+y^2)^2}.$$

(2) Let $f(x,y) = 2x^3 + 3xy^2$. Verify that:

$$f_{xx} = 12x;\ f_{yy} = 6x;\ \text{and}$$

$$f_{xy} = f_{yx} = 6y.$$

Remark:
If f_{xy} and f_{yx} are continuous, then $f_{xy} = f_{yx}$, and the order of differentiation is immaterial.

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Implicit Differentiation

Suppose $u = f(x, y)$ is a continuous function defined in a region $\mathbb{R}$ of the xy-plane, and both f_x and f_y are continuous in this region. If x and y are differentiable functions of a variable t so that

$$x = x(t) \text{ and } y = y(t),$$

then $u = u(t)$, and

$$\frac{du}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt}$$

A special case of this arises when y is itself a function of x (i.e; $t = x$). Consequently, $u = u(x)$, and

$$\frac{du}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dx} \quad \text{------} (*)$$

For example, let $u = f(x, y) = \arctan(x/y)$ and $y = \sin x$. Then

$$\frac{du}{dx} = \frac{y}{x^2 + y^2} - \frac{x}{x^2 + y^2} \cos x$$

$$= \frac{\sin x - x\cos x}{x^2 + \sin^2 x}$$

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<u>Remark</u>: An alternative way of obtaining $\frac{dy}{dx}$ is to differentiate $u = \arctan\left(\frac{x}{\sin x}\right)$ with respect to x.

When y is defined as a function of x by the equation

$$u = f(x, y) = 0,$$

y is called an implicit function of x.

Since $u = 0$, then $\frac{du}{dx} = 0$ and from (*), we obtain

$$\frac{dy}{dx} = -\frac{\left(\frac{\partial f}{\partial x}\right)}{\left(\frac{\partial f}{\partial y}\right)}$$

<u>Example</u>: Find the gradient of the conic:

$$f(x, y) = ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0,$$

where a, h, b, g, f and c are constants

$$\text{Gradient} = \frac{dy}{dx} = -\frac{\left(\frac{\partial f}{\partial x}\right)}{\left(\frac{\partial f}{\partial y}\right)} = \frac{-2(ax + hy + g)}{2(by + hx + f)}$$

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Consider the equations:

$F(x, y, z) = 0$ and $G(x, y, z) = 0$, where F and G are differentiable functions of x, y and z.
Since $F = 0$ and $G = 0$, then the total derivatives of F and G are identically zero.
Suppose y and z are functions of x, then

$$\frac{dF}{dx} = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} \cdot \frac{dy}{dx} + \frac{\partial F}{\partial z} \cdot \frac{dz}{dx} = 0$$

and

$$\frac{dG}{dx} = \frac{\partial G}{\partial x} + \frac{\partial G}{\partial y} \cdot \frac{dy}{dx} + \frac{\partial G}{\partial z} \cdot \frac{dz}{dx} = 0$$

Elliminating $\frac{dz}{dx}$ from the two equations yields

$$\frac{dy}{dx} = \frac{-\left(\frac{\partial F}{\partial x} \cdot \frac{\partial G}{\partial z} - \frac{\partial F}{\partial z} \cdot \frac{\partial G}{\partial x}\right)}{\left(\frac{\partial F}{\partial y} \cdot \frac{\partial G}{\partial z} - \frac{\partial F}{\partial z} \cdot \frac{\partial G}{\partial y}\right)}$$

Similarly, elliminating $\frac{dy}{dx}$, we have,

$$\frac{dz}{dx} = \frac{\left(\frac{\partial F}{\partial x}\cdot\frac{\partial G}{\partial y} - \frac{\partial F}{\partial y}\cdot\frac{\partial G}{\partial x}\right)}{\left(\frac{\partial F}{\partial y}\cdot\frac{\partial G}{\partial z} - \frac{\partial F}{\partial z}\cdot\frac{\partial G}{\partial y}\right)}$$

Example: Let $F(x,y,z) = x^2+y^2+z^2$
and $G(x,y,z) = x^2-y^2+2z^2$

Then

$$\frac{\partial F}{\partial x} = 2x,\ \frac{\partial F}{\partial y} = 2y,\ \frac{\partial F}{\partial z} = 2z,$$

$$\frac{\partial G}{\partial x} = 2x,\ \frac{\partial G}{\partial y} = -2y,\ \frac{\partial G}{\partial z} = 4z.$$

Hence, $\frac{dy}{dx} = \frac{-x}{3y}$ and $\frac{dz}{dx} = -\frac{2x}{3x}$.

(2) Let $z = e^{xy^2}$, $x = t\cos t$, $y = t\sin t$.
Find $\frac{dz}{dt}$ at $t = \pi/2$.

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

$$= (y^2 e^{xy^2})(-t\sin t + \cos t) + 2xye^{xy^2}(t\cos t + \sin t)$$

At $t = \pi/2$, $x = 0$, $y = \pi/2$. Then

$$\left.\frac{dz}{dt}\right|_{t=\pi/2} = (\pi^2/4)(-\pi/2) + (0)(1) = -\frac{\pi^3}{8}$$

A function $f(x,y)$ is said to be homogeneous of degree m if

$$f(kx, ky) = k^m f(x,y),$$

where k is a constant.

Example (1) The function
$$f(x,y) = x^3 + 4xy^2 - 3y^2$$
is homogeneous of degree 3 since
$$(kx)^3 + 4(kx)(ky)^3 - 3(ky)^3 = k^3(x^3 + 4xy^2 - 3y^3)$$
$$= k^3 f(x,y).$$

(2) $f(x,y) = \dfrac{x^2+y^2}{4xy} + \dfrac{y}{x}\sin\left(\dfrac{x}{y}\right)$ is homogeneous of degree 0.

Euler's Theorem:

If $u = f(x_1, x_2, \cdots, x_n)$ is a homogeneous differentiable function of degree m in the independent variables $x_1, x_2, \cdots, x_n$, then
$$x_1 \frac{\partial f}{\partial x_1} + x_2 \frac{\partial f}{\partial x_2} + \cdots + x_n \frac{\partial f}{\partial x_n} = mf.$$

Example: Consider the function
$$f(x,y) = x^3 + 4xy^2 - 3y^3.$$

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Question........

Verify that it is homogeneous of degree 3. Hence, by Euler's Theorem,

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = 3f.$$

[Now verify that this is correct by computing $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ first, then simplify $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y}$.]

Exercise

Evaluate $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y}$ when

(a) $f(x,y) = xy - \frac{1}{x+y}$,

(b) $f(x,y) = \log_e\left(\frac{y}{x}\right)$,

(c) $f(x,y) = e^{xy}$

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Taylor's Theorem for Functions of Two Independent Variables.

If $f(x,y)$ is defined in a region $\mathcal{R}$ of the xy-plane and all its partial derivatives of orders up to and including the $(n+1)$th are continuous in $\mathcal{R}$, then for any point (a,b) in this region,

$$f(a+h, b+k) = f(a,b) + \bar{D}f(a,b) + \frac{1}{2!}\bar{D}^2 f(a,b) + \cdots + \frac{1}{n!}\bar{D}^n f(a,b) + E_n,$$

where $\bar{D}$ is the differential operator defined by

$$\bar{D} = h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y},$$

and

$\bar{D}^m f(a,b)$ means $\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^m f(x,y)$ evaluated at the point (a,b).

The error term E_n is given by

$$E_n = \frac{1}{(n+1)!}\bar{D}^{n+1} f(a+$$

*We assume that $\lim_{n\to\infty} E_n = 0$.

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An alternative form of Taylor's series may be obtained by putting

$$h = x - a \text{ and } k = y - b.$$

Then

$$f(x,y) = f(a,b) + [(x-a)f_x(a,b) + (y-b)f_y(a,b)]$$
$$+ \frac{1}{2!}\{(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b)f_{xy}(a,b)$$
$$+ (y-b)^2 f_{yy}(a,b)\}$$
$$+ \cdots,$$

which is Taylor's expansion of $f(x,y)$ about the point (a,b).

<u>Note</u> that

$$\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 = h^2 f_{xx} + 2hk f_{xy} + k^2 f_{yy};$$

we formally expand

$$\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n$$

by the binomial theorem.

<u>Example</u>: Expand the function

$$f(x,y) = \sin(xy)$$

about the point $(1, \pi/3)$, neglecting terms of degree ≥ 3.

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$f(1, \pi/3) = \sin(\pi/3) = \sqrt{3}/2,$

$f_x(x, y) = y\cos(xy) \Rightarrow f_x(1, \pi/3) = \pi/6,$

$f_y(x, y) = x\cos(xy) \Rightarrow f_y(1, \pi/3) = 1/2$

$f_{xx}(x, y) = -y^2\sin(xy) \Rightarrow f_{xx}(1, \pi/3) = -\frac{\pi^2\sqrt{3}}{18}$

$f_{xy}(x, y) = -xy\sin(xy) + \cos xy$

$\Rightarrow f_{xy}(1, \pi/3) = \frac{-\pi\sqrt{3}}{6} + \frac{1}{2},$

$f_{yy}(x, y) = -x^2\sin(xy) \Rightarrow f_{yy}(1, \pi/3) = -\frac{\sqrt{3}}{2}.$

We stop the computation here, since we are to neglect terms of degree ≥ 3.

Now substituting the values into the Taylor's expansion, we have

$$\sin(xy) \approx \frac{\sqrt{3}}{2} + \left\{(x-1)\left(\frac{\pi}{6}\right) + \left(y - \frac{\pi}{3}\right)\left(\frac{1}{2}\right)\right\} + \frac{1}{2!}\left\{(x-1)^2\left(-\frac{\pi^2\sqrt{3}}{18}\right) + 2(x-1)\left(y-\frac{\pi}{3}\right)\left(-\frac{\pi\sqrt{3}}{6} + \frac{1}{2}\right) + \left(y - \pi/3\right)^2\left(-\frac{\sqrt{3}}{2}\right)\right\}.$$

Students may simplify further ~~to obtain~~

Exercise (1) Expand $f(x,y) = \sin(xy)$ in powers of $x-1$ and $y-\pi/2$ to second-degree terms.

[Ans: $1 - \frac{1}{8}\pi^2(x-1)^2 - \frac{\pi}{2}(x-1)(y-\frac{\pi}{2}) - (y-\frac{\pi}{2})^2$]

(2) Expand x^2+3y-2 in powers of $x-1$ and $y+2$.

[Hint: Use Taylor's with $h = x-1$ and $k = y-(-2)$]

{Ans: $-10-4(x-1)+4(y+2)-2(x-1)^2 + 2(x-1)(y+2)+(x-1)^2(y+2)$}

(3) Expand $f(x,y) = e^{xy}$ to three terms about the point $x=2$, $y=3$.

Ans: $e^6\{1+3(x-2)+2(y-3)+\frac{9}{2}(x-2)^2 + 7(x-2)(y-3)+2(y-3)^2+\cdots\}$.

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Maxima and Minima of Functions of Two Variables.

A function $f(x,y)$ is said to have a maximum value at a point $(x,y)=(a,b)$ if

$$f(a+h,b+k)-f(a,b)<0,$$

where h and k are small arbitrary quantities.

Similarly $f(x,y)$ is said to have a minimum at $(x,y)=(a,b)$ if

$$f(a+h,b+k)-f(a,b)>0.$$

For maximum/minimum to occur at any point, it is required that

$$\frac{\partial f}{\partial x}=0 \text{ and } \frac{\partial f}{\partial y}=0$$

at the point.

From the Taylor's expansion

$$f(a+h,b+k)=f(a,b)+\bar{D}f(a,b)+\frac{1}{2!}\bar{D}^2f(a,b)+\cdots,$$

where $\bar{D}=\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)$,
observing that, at a critical point

(a,b) where $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$, we have

$$\bar{D}f = h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y} = 0, \text{ and}$$

$$\bar{D}^2 f = h^2\frac{\partial^2 f}{\partial x^2} + 2hk\frac{\partial^2 f}{\partial x \partial y} + k^2\frac{\partial^2 f}{\partial y^2},$$

now neglecting terms of order h^3, k^3 and higher, we have

$$f(a+h, b+k) - f(a,b) = \frac{1}{2}\{h^2 f_{xx}(a,b) + 2hk f_{xy}(a,b) + k^2 f_{yy}(a,b)\}.$$

By adding $k^2 f_{xy}^2(a,b) - k^2 f_{xy}^2(a,b)$, and multiplying $\frac{f_{xx}(a,b)}{f_{xx}(a,b)}$ into the right hand side of the equation we obtain

$$f(a+h, b+k) - f(a,b) = \frac{1}{2f_{xx}(a,b)}\{[hf_{xx}(a,b) + kf_{xy}(a,b)]^2 - k^2[f_{xy}^2(a,b) - f_{xx}(a,b)f_{yy}(a,b)]\}$$

If $\Delta \equiv f_{xy}^2(a,b) - f_{xx}(a,b)f_{yy}(a,b)$, Question..................

Then we deduce that (a,b) is a maximum if

$$\Delta < 0 \text{ and } f_{xx}(a,b) < 0,$$

and a minimum if

$$\Delta < 0 \text{ and } f_{xx}(a,b) > 0.$$

Note

1) We can replace $f_{xx}(a,b)$ with $f_{yy}(a,b)$ in the above.

2) When $\Delta > 0$, the stationary point is called a saddle point. Such a point is neither maximum nor minimum.

3) When $\Delta = 0$, further investigation is necessary to determine the nature of the stationary point.

Examples

(1) Find the nature of the stationary points of the function

$$f(x,y) = x^4 + 4x^2y^2 - 2x^2 + 2y^2 - 1$$

The necessary conditions $\frac{\partial f}{\partial x} = 0$ and

Question

$\frac{\partial f}{\partial y} = 0$ for existence of stationary points yield the following two equations

$$4x(x^2+2y^2-1) = 0 \quad \text{———} \quad (i)$$

and

$$4y(1+2x^2) = 0 \quad \text{———} \quad (ii)$$

The only condition for (ii) to be true in $\mathbb{R}$ is that $y=0$.
Now, substituting $y=0$ into (i),

$$x(x^2-1) = 0 \Rightarrow x = 0, \pm 1.$$

Thus, the stationary points of $f(x,y)$ are $(0,0)$, $(1,0)$ and $(-1,0)$.

Meanwhile, $f_{xx} = 12x^2 + 8y^2 - 4$,
$f_{yy} = 8x^2 + 4$,
and $f_{xy} = 16xy$.

Therefore, at the point $(0,0)$, $f_{xx} = -4$, $f_{yy} = 4$ and $f_{xy} = 0$, giving us

$$\Delta = f_{xy}^2(0,0) - f_{xx}(0,0)f_{yy}(0,0) = 16 > 0$$

The point $(0,0)$ is therefore a saddle point

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At (1, 0), $f_{xx} = 8$, $f_{yy} = 12$, $f_{xy} = 0$.
Thus, $\Delta = -96 < 0$
Therefore, the point (1, 0) is a minimum
Similarly, the point (−1, 0) is found to be a minimum.

Hence, the function $f(x, y)$ has two minima (at (1, 0) and (−1, 0)), and one saddle point (at (0, 0)).
Furthermore,

$f(x, y) = -2$ at both minima, and
$f(x, y) = -1$ at the saddle point.

(2) Find the maxima and minima of
$$f(x,y) = x^3 + y^3 - 3x - 12y + 20.$$

The necessary conditions for critical points,
$\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$ yield.

$f_x = 3x^2 - 3 = 0$ and $f_y = 3y^2 - 12 = 0$.

Thus, the critical points are:

$P_1(1, 2)$, $P_2(-1, 2)$, $P_3(1, -2)$ and $P_4(-1, -2)$.

$f_{xx} = 6x$, $f_{yy} = 6y$, $f_{xy} = 0$. Then $\Delta = -36xy$.

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Question

At $P_1(1,2)$, $\Delta \leq 0$ and $f_{xx} > 0$, hence P_1 is a minimum point

~~At $P_2(-1,2)$, $\Delta \geq 0$ and f_{xx}~~

At $P_2(-1,2)$, $\Delta > 0$. P_2 is a saddle point.

At $P_3(1,-2)$, $\Delta > 0$. P_3 is a saddle point

At $P_4(-1,-2)$, $\Delta < 0$ and $f_{xx} < 0$, so P_4 is a maximum point.

Further, the minimum value of $f(x,y)$ occuring at $P_1(1,2)$ is 2, The maximum value occuring at $P_4(-1,-2)$ is 36.

Exercise:

1) A rectangular box, open at the top, is to have a volume $32\ ft^3$. What must be the dimension so that the total surface is a minimum?
[4ft × 4ft × 2ft]

2) Determine the nature of the stationary points of the surface

$$z = x^3 + xy + y^2$$

[(0,0) saddle pt, $(\frac{1}{6}, -\frac{1}{12})$ minimum]

Lagrange Multiplier

Suppose $f(x,y)$ is to be examined for stationary points subject to the constraint

$$g(x,y) = 0$$

We introduce a parameter λ such that

$$\frac{\partial f}{\partial x} + \lambda \frac{\partial g}{\partial x} = 0 \text{ and } \frac{\partial f}{\partial y} + \lambda \frac{\partial g}{\partial y} = 0$$

The three equations above are sufficient to determine the stationary points and the value of the multiplier λ.

Examples

1) Find the maximum distance from the origin $(0,0)$ to the curve

$$3x^2 + 3y^2 + 4xy - 2 = 0.$$

The problem here is to find the maximum value of the distance d, where

$$d^2 = f(x,y) = x^2 + y^2$$

subject to the constraint

$$g(x,y) = 3x^2 + 3y^2 + 4xy - 2 = 0.$$

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Question

We are to solve the Lagrange equations

$$2x + \lambda(6x + 4y) = 0 \quad \text{——— (1)}$$

and $$2y + \lambda(6y + 4x) = 0 \quad \text{——— (2)}$$

together with $g(x, y) = 0$ ——— (3)

Now, $(1)y \Rightarrow 2xy + \lambda(6xy + 4y^2) = 0$

$(2)x \Rightarrow 2xy + \lambda(6xy + 4x^2) = 0$

$(1)y - (2)x \Rightarrow 4\lambda(y^2 - x^2) = 0$

$\Rightarrow y = \pm x$

When $y = x$, (3) yields $10x^2 - 2 = 0$

$\Rightarrow x = \pm \frac{1}{\sqrt{5}}$.

When $y = -x$, (3) yields $2x^2 - 2 = 0$

$\Rightarrow x = \pm 1$

Thus, the stationary points are:

$\left(\frac{1}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right), \left(-\frac{1}{\sqrt{5}}, -\frac{1}{\sqrt{5}}\right), (1, -1), (-1, 1)$.

For the first two points, $d^2 = 2/5$, while for the last two points, $d^2 = 2$. Hence the maximum distance from the origin to the curve is $d = \sqrt{2}$.

Exercises

1) Find the shortest distance

Question

from the origin to the hyperbola

$$x^2 + 8xy + 7y^2 = 225 \; , \; z = 0 .$$

[Ans: $d = 5$]

2) Find the stationary points of the function

$$V = x^2 + y^2 + z ,$$

subject to the condition

$$x^2 - z^2 = 1 .$$

$[(\sqrt{5}/2, 0, -\tfrac{1}{2}), (-\sqrt{5}/2, 0, -\tfrac{1}{2})]$

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Question

Jacobian Determinant

If $F(u,v)$ and $G(u,v)$ are differentiable in a region $\mathcal{R}$, the Jacobian of F and G with respect to u and v is the second order functional determinant defined by

$$\frac{\partial(F,G)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial F}{\partial u} & \frac{\partial F}{\partial v} \\ \frac{\partial G}{\partial u} & \frac{\partial G}{\partial v} \end{vmatrix} = \begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}$$

Similarly, the third order determinant

$$\frac{\partial(F,G,H)}{\partial(u,v,w)} = \begin{vmatrix} F_u & F_v & F_w \\ G_u & G_v & G_w \\ H_u & H_v & H_w \end{vmatrix}$$

is called the Jacobian of F, G and H with respect to u, v and w.

Jacobians are useful in obtaining partial derivatives of implicit functions. For example, given the equations

$$F(x,y,u,v) = 0, \quad G(x,y,u,v) = 0,$$

if u and v are functions of x and y, we have

$$\frac{\partial u}{\partial x} = -\frac{\frac{\partial(F,G)}{\partial(x,v)}}{\frac{\partial(F,G)}{\partial(u,v)}} = \frac{-\begin{vmatrix} F_x & F_v \\ G_x & G_v \end{vmatrix}}{\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}},$$

$$\frac{\partial u}{\partial y} = -\frac{\frac{\partial(F,G)}{\partial(y,v)}}{\frac{\partial(F,G)}{\partial(u,v)}} = \frac{-\begin{vmatrix} F_y & F_v \\ G_y & G_v \end{vmatrix}}{\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}},$$

$$\frac{\partial v}{\partial x} = -\frac{\frac{\partial(F,G)}{\partial(u,x)}}{\frac{\partial(F,G)}{\partial(u,v)}} = -\frac{\begin{vmatrix} F_u & F_x \\ G_u & G_x \end{vmatrix}}{\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}},$$

$$\frac{\partial v}{\partial y} = -\frac{\frac{\partial(F,G)}{\partial(u,y)}}{\frac{\partial(F,G)}{\partial(u,v)}} = -\frac{\begin{vmatrix} F_u & F_y \\ G_u & G_y \end{vmatrix}}{\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}}.$$

Example

If $u^2 - v = 3x + y$ and $u - 2v^2 = x - 2y$, find (a) $\frac{\partial u}{\partial x}$, (b) $\frac{\partial v}{\partial x}$, (c) $\frac{\partial u}{\partial y}$ and

(d) $\frac{\partial v}{\partial y}$.

The given equations F and G are:

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Question

$F(x, y, u, v) = u^2 - v - 3x - y = 0$;
$G(x, y, u, v) = u - 2v^2 - x + 2y = 0$.

Then,

$$\frac{\partial u}{\partial x} = -\frac{\frac{\partial(F, G)}{\partial(x, v)}}{\frac{\partial(F, G)}{\partial(u, v)}} = -\frac{\begin{vmatrix} F_x & F_v \\ G_x & G_v \end{vmatrix}}{\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}}$$

$$= -\frac{\begin{vmatrix} -3 & -1 \\ -1 & -4v \end{vmatrix}}{\begin{vmatrix} 2u & -1 \\ 1 & -4v \end{vmatrix}}$$

$$= \frac{1 - 12v}{1 - 8uv}, \quad 1 - 8uv \neq 0.$$

The remaining part ((b), (c), (d)) is left as an easy exercise.

$$\left[\frac{\partial v}{\partial x} = \frac{2u - 3}{1 - 8uv}, \quad \frac{\partial u}{\partial y} = \frac{-2 - 4u}{1 - 8uv} \right.$$
$$\left. \frac{\partial v}{\partial y} = \frac{-4u - 1}{1 - 8uv}. \right]$$

MTH 201: Numerical Methods

March 1, 2022

0.1 Solution of Equations in One Variable

Suppose that f is a real-valued function, defined and continuous on a bounded closed interval $[a, b]$ of the real line. It will be tacitly assumed that $a < b$, so that the interval is nonempty. We wish to find a real number $\xi \in [a, b]$ such that $f(\xi) = 0$. If such ξ exists, it is called a **solution** to the equation $f(x) = 0$. So, finding the roots of $f(x)$ means solving the equation

$$f(x) = 0. \tag{1}$$

If $f(x)$ is a quadratic, cubic or a biquadratic expression, then algebraic formulae are available for expressing the roots in terms of the coefficients. On the other hand, when $f(x)$ is a polynomial of higher degree or an expression involving transcendental functions, algebraic methods are not available, and recourse must be taken to find the roots by approximate methods.
Note: A non-algebraic function is called a transcendental function, e.g. $f(x) = \ln x^3 - 0.7$, $\phi(x) = e^{-0.5x} - 5x$, $\psi(x) = \sin^2(x) - x^2 - 2$, etc.

Definition 1. Root of an Equation, Zero of a Function: *Assume that $f(x)$ is a continuous function. Any number ξ for which $f(\xi) = 0$ is called* **a root of the equation** $f(x) = 0$. *Also, we say ξ is* **a zero of the function** $f(x)$.

Theorem 1. *Let f be a real-valued function, defined and continuous on a bounded closed interval $[a, b]$ of the real line. Assume, further, that $f(a)f(b) \le 0$; then, there exists ξ in $[a, b]$ such that $f(\xi) = 0$.*

The roots of equation (1) may be either real or complex. Hundreds of methods are available for locating zeros of functions, and two of the most useful have been selected for study here: the **bisection** and **Newton's methods.**

0.1.1 Bisection Method

Let f be a function that has values of opposite sign at the two ends of an interval. Suppose also that f is continuous on that interval. To fix the notation, let $a < b$ and $f(a)f(b) < 0$. It then follows that f has a root in the interval (a, b). In other words, there must exist a number p that satisfies the two conditions $a < p < b$ and $f(p) = 0$. How is this conclusion reached? One must recall the **Intermediate Value Theorem**. "If x traverses an interval $[a, b]$, then the values of $f(x)$ completely fill out the interval between $f(a)$ and $f(b)$." No intermediate values can be skipped. Hence, a specific function f must

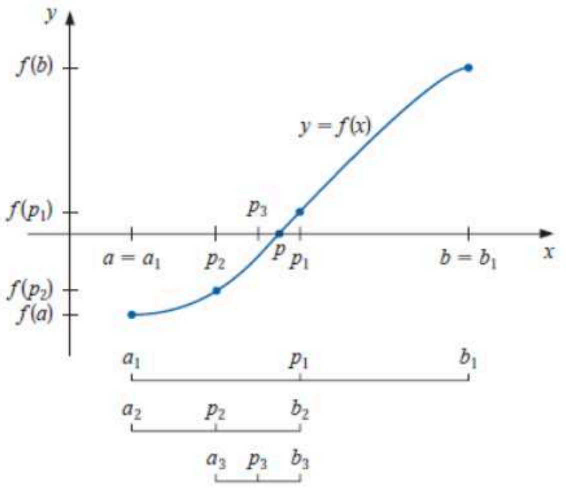

Figure 1:

take on the value zero somewhere in the interval (a, b) because $f(a)$ and $f(b)$ are of opposite signs.

The Bisection method calls for a repeated halving (or bisecting) of subintervals of $[a, b]$ and, at each step, locating the half containing p.

To begin, set $a_1 = a$ and $b_1 = b$ and let p_1 be the midpoint of $[a, b]$; that is,

$$p_1 = a_1 + \frac{b_1 - a_1}{2} = \frac{a_1 + b_1}{2}.$$

- If $f(p_1) = 0$, then $p = p_1$, and we are done.
- If $f(p_1) \neq 0$, then $f(p_1)$ has the same sign as either $f(a_1)$ or $f(b_1)$.
- If $f(p_1)$ and $f(a_1)$ have the same sign, $p \in (p_1, b_1)$. Set $a_2 = p_1$ and $b_2 = b_1$.
- If $f(p_1)$ and $f(a_1)$ have opposite signs, $p \in (a_1, p_1)$. Set $a_2 = a_1$ and $b_2 = p_1$.

Then reapply the process to the interval $[a_2, b_2]$.

Example 1. *Show that $f(x) = x^3 + 4x^2 - 10 = 0$ has a root in $[1, 2]$, and use the Bisection method to determine an approximation to the root that is accurate to at least within 10^{-4}.*

Solution: Because $f(1) = -5$ and $f(2) = 14$, the Intermediate Value Theorem ensures that this is continuous in $[1, 2]$.

For the first iteration, we use the fact that at the midpoint of $[1,2]$ we have $f(1.5) = 2.375 > 0$. This indicates that we should select the interval $[1,1.5]$ for our second iteration. Then we find that $f(1.25) = -1.796875$, so our new interval becomes $[1.25,1.5]$, whose midpoint is 1.375. Continuing in this manner gives the values in Table 1.
After 13 iterations, $p_{13} = 1.365112305$ approximates the root p with an error

$$|p - p_{13}| < |b_{14} - a_{14}| = |1.365234375 - 1.365112305| = 0.000122070.$$

Since $|a_{14}| < |p|$, we have

$$\frac{|p - p_{13}|}{|p|} < \frac{|b_{14} - a_{14}|}{|a_{14}|} \leq 9.0 \times 10^{-5},$$

so the approximation is correct to at least within 10^{-4}. The correct value of p to nine decimal places is $p = 1.365230013$. **Note** that p_9 is closer to p than the final approximation p_{13}.

Theorem 2. *Suppose that $f \in C[a,b]$ and $f(a) \cdot f(b) < 0$. The Bisection method generates a sequence $\{p_n\}_{n=1}^{\infty}$ approximating a zero p of f with*

$$\left|p_n - p\right| \leq \frac{b-a}{2^n}, \quad when \quad n \geq 1. \tag{2}$$

Example 2. *Determine the number of iterations necessary to solve $f(x) = x^3 + 4x^2 - 10 = 0$ with accuracy 10^{-3} using $a_1 = 1$ and $b_1 = 2$.*

Solution: From equation (2), we have

$$\left|p_n - p\right| \leq 2^{-n}(b-a) = 2^{-n} < 10^{-3}.$$

We will use base-10 logarithms because the tolerance is given as a power of 10.
Thus

$$2^{-n} < 10^{-3}$$

implies that

$$\log_{10} 2^{-n} < \log_{10} 10^{-3} = -3.$$

We then have

$$-n \log_{10} 2 < -3$$

from which we obtain

$$n > \frac{3}{\log_{10} 2} \approx 9.96.$$

Hence, ten iterations will ensure an approximation accurate to within 10^{-3}.

0.1.2 Advantages of Bisection Method

i. Provided the function is continuous on an interval $[a, b]$ with $f(a) \cdot f(b) < 0$, bisection method is guaranteed to work.

ii. The number of iterations needed to achieve a specific accuracy is known in advance.

0.1.3 Disadvantages of Bisection Method

i. The method is slow to converge.

ii. The errors in p_n and in $f(p_n)$ do not necessarily decrease between iterations.

EXERCISE 1

(a) Use the Bisection method to find p_3 for $f(x) = \sqrt{x} - \cos(x)$ on $[0, 1]$

(b) Use the Bisection method to find solutions accurate to within 10^{-5} for the following problems:

i. $x - 2^{-x} = 0 \quad for \quad 0 \leq x \leq 1$

ii. $e^x - x^2 + 3x - 2 = 0 \quad for \quad 0 \leq x \leq 1$

iii. $2x\cos(2x) - (x+1)^2 = 0 \quad for \quad -3 \leq x \leq -2$

(c) If $a = 0.1$ and $b = 1.0$, how many steps of the bisection method are needed to determine the root with an error of at most $\frac{1}{2} \times 10^{-8}$?

(d) Use Theorem 2 to find a bound for the number of iterations needed to achieve an approximation with accuracy 10^{-4} to the solution of $x^3 - x - 1 = 0$ lying in the interval $[1, 2]$. Find an approximation to the root with this degree of accuracy.

(e) Consider the bisection method with the initial interval $[a_0, b_0]$. Show that after ten steps with this method,

$$\left|\frac{1}{2}\Big(a_{10} + b_{10}\Big) - \frac{1}{2}\Big(a_9 + b_9\Big)\right| = 2^{-11}\big(b_0 - a_0\big).$$

(f) The function $f(x) = \sin(x)$ satisfies $f(-\pi/2) = -1$ and $f(\pi/2) = 1$. Using bisection method, how many iterations are needed to find an interval of length at most 10^{-4} which contains a root of the function?

0.1.4 Newton's Method

The procedure known as Newton's method is also called the **Newton-Raphson iteration.** The Newton-Raphson method is one of the most useful and best known algorithms that relies on the continuity of $f'(x)$ and $f''(x)$.

0.1.5 Interpretations of Newton's Method

In Newton's method, it is assumed at once that the function f is differentiable. This implies that the graph of f has a definite slope at each point and hence a unique tangent line.

At a certain point $(x_0, f(x_0))$ on the graph of f, there is a tangent, which is a rather good approximation to the curve in the vicinity of that point. Analytically, it means that the linear function

$$l(x) = f'(x_0)(x - x_0) + f(x_0)$$

is close to the given function f near x_0. At x_0, the two functions l and f agree. We take the zero of l as an approximation to the zero of f. The zero of l is easily found:

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}.$$

Thus, starting with point x_0 (which we may interpret as an approximation to the root sought), we pass to a new point x_1 obtained from the preceding formula. Naturally, the process can be repeated (iterated) to produce a sequence of points:

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}, \quad x_3 = x_2 - \frac{f(x_2)}{f'(x_2)},$$

etc. Under favourable conditions, the sequence of points will approach a zero of f.

The geometry of Newton's method is shown in Figure 1. The line $y = l(x)$ is tangent to the curve $y = f(x)$. It intersects the x- axis at a point x_1. The slope of $l(x)$ is $f'(x_0)$.

There are other ways of interpreting Newton's method. Suppose again that x_0 is an initial approximation to a root of f and let $x_1 = x_0 + h$ be the correct root so that $f(x_1) = 0$.
Expanding $f(x_0 + h)$ by Taylor series, we obtain

$$f(x_0) + hf'(x_0) + \frac{h^2}{2}f''(x_0) + \cdots = 0$$

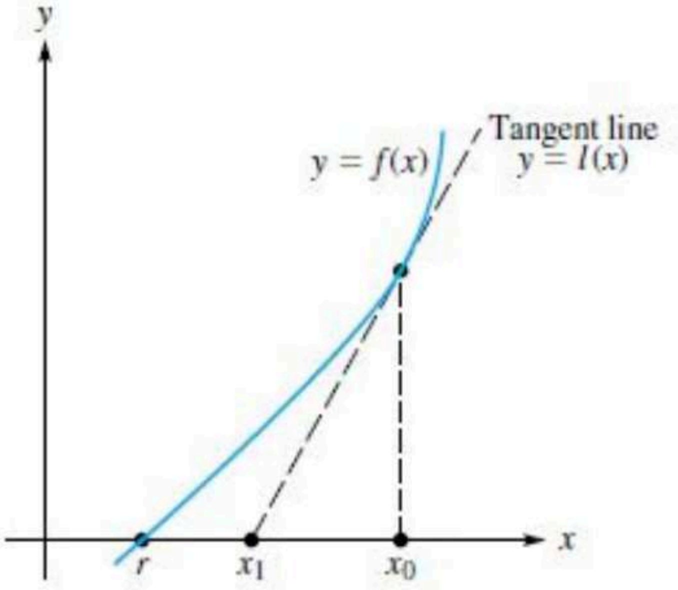

Figure 2:

Neglecting the second and higher order derivatives, we have

$$f(x_0) + hf'(x_0) = 0,$$

which gives

$$h = -\frac{f(x_0)}{f'(x_0)}.$$

A better approximation than x_0 is therefore, given by x_1, where

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}.$$

Successive approximations are given by $x_2, x_3, \cdots, x_{n+1}$, where

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \tag{3}$$

which is the Newton-Raphson formula.

Example 3. *Perform four iterations of the Newton's method to find the smallest positive root of the equation $f(x) = x^3 - 5x + 1 = 0$.*

Solution:

$$f(x) = x^3 - 5x + 1 = 0,\ therefore \quad f'(x) = 3x^2 - 5, \quad f(0) = 1, \quad and \quad f(1) = -3.$$

Since, $f(0) \cdot f(1) < 0$, the smallest positive root lies in the interval $(0, 1)$. Applying the Newton's method, we obtain

$$x_{n+1} = x_n - \frac{x_n^3 - 5x_n + 1}{3x_n^2 - 5} = \frac{2x_n^3 - 1}{3x_n^2 - 5}, \quad n = 0, 1, 2, 3.$$

The initial approximation x_0 is taken to be $\frac{0+1}{2} = 0.5$. Thus we have the following

$$x_1 = \frac{2x_0^3 - 1}{3x_0^2 - 5} = \frac{2(0.5)^3 - 1}{3(0.5)^2 - 5} = 0.176471,$$

$$x_2 = \frac{2x_1^3 - 1}{3x_1^2 - 5} = \frac{2(0.176471)^3 - 1}{3(0.176471)^2 - 5} = 0.201568,$$

$$x_3 = \frac{2x_2^3 - 1}{3x_2^2 - 5} = \frac{2(0.201568)^3 - 1}{3(0.201568)^2 - 5} = 0.201640,$$

$$x_4 = \frac{2x_3^3 - 1}{3x_3^2 - 5} = \frac{2(0.201640)^3 - 1}{3(0.201640)^2 - 5} = 0.201640.$$

Therefore, the root correct to six decimal places is $x = 0.201640$.

Example 4. *Show that the equation*

$$f(x) = cos\left(\frac{\pi(x+1)}{8}\right) + 0.148x - 0.9062 = 0$$

has one root in the interval $(-1, 0)$ and one in $(0, 1)$. Calculate the negative root correct to 4 decimal places.

Solution: From the given function, we have

$$f(-1) = -0.0542, \quad f(0) = 0.0177, \quad f(1) = -0.0511.$$

Hence, one root lies in the interval $(-1, 0)$ and one root in the interval $(0, 1)$. Using the formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \ n = 0, 1, \cdots$$

where

$$f'(x) = -\left(\frac{\pi}{8}\right) \sin\left(\frac{\pi(x+1)}{8}\right) + 0.148.$$

With $x_0 = -0.5$, we obtain the following sequence of iterates:

$$x_1 = -0.508199, \quad x_2 = -0.508129, \quad x_3 = -0.508129.$$

Hence, the root correct to four decimal places is -0.5081.

0.1.6 Advantages of Newton's Method

(i) It is fast

(ii) It can be extended to multidimensional problems.

0.1.7 Disadvantages of Newton's Method

(i) It requires evaluating the derivative, $f'(x)$, at each iteration.

(ii) It can fail if the initial guess is not sufficiently close to the solution, particularly if $f'(x) = 0$.

EXERCISE 2

(a) Find all the roots of $\cos(x) - x^2 - x = 0$ to five decimal places.

(b) Verify that when Newton's method is used to compute $\sqrt{R}$, the sequence of iterates is defined by

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{R}{x_n}\right).$$

(c) Establish Newton's iterative scheme in simplified form, not involving the reciprocal of x for the function $f(x) = xR - x^{-1}$. Carry out three steps of this procedure using $R = 4$ and $x_0 = -1$.

(d) Use Newton's method to find all roots of the following equations, correct to six decimal places
(i) $3\cos(x) = x + 1$ **(ii)** $\cos(x) = \sqrt{x}$ **(iii)** $\sin(x) = x^2 - 2$ **(iv)** $\cos(x^2 - x) = x^4$.

(e) Find all positive roots to the equation

$$10\int_0^x e^{-x^2}\,dt = 1,$$

correct to six decimal places.

(f) The function described by $f(x) = \ln(x^2 + 1) - e^{0.4x}\cos(\pi x)$ has an infinite number of zeros. (i) Determine, within 10^{-6}, the only negative zero. (ii) Determine, within 10^{-6}, the four smallest positive zeros.

0.2 Numerical Integration

Numerical integration is a primary tool used by engineers and scientists to obtain approximate answers for definite integrals that cannot be solved analytically.

0.2.1 Newton-Cotes Quadrature Formula

Newton-Cotes quadrature formulae for approximating $\int_a^b f(x)dx$, are obtained by approximating the function of integration $f(x)$ by interpolating polynomials. The rules are closed when they involve function values at the ends of the interval of integration. Otherwise, they are said to be open.
The general problem of numerical integration may be stated as follows:
Given a set of data points $(x_0, y_0), (x_1, y_1), \cdots, (x_n, y_n)$ of a function $y = f(x)$, where $f(x)$ is not known explicitly, it is required to compute the value of definite integral

$$I = \int_a^b y dx.$$

Let the interval $[a, b]$ be divided into n equal subintervals such that $a = x_0 < x_1 < x_2 < \cdots < x_n = b$. Clearly, $x_n = x_0 + nh$. Hence the integral becomes

$$I = \int_{x_0}^{x_n} y dx$$

Approximating y by Newton's forward difference formula, we obtain

$$I = \int_{x_0}^{x_n} \left[y_0 + r\Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_0 + \frac{r(r-1)(r-2)}{3!}\Delta^3 y_0 + \cdots \right] dx$$

Since $x = x_0 + rh$, $dx = hdr$ and hence the above integral becomes

$$I = h\int_0^n \left[y_0 + r\Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_0 + \frac{r(r-1)(r-2)}{3!}\Delta^3 y_0 + \cdots \right] dr$$

$$= h\left[ry_0 + \frac{r^2}{2}\Delta y_0 + \frac{1}{2}\left(\frac{r^3}{3} - \frac{r^2}{3}\right)\Delta^2 y_0 + \frac{1}{6}\left(\frac{r^4}{4} - r^3 + r^2\right)\Delta^3 y_0 + \cdots \right]_0^n$$

which on simplification gives

$$\int_{x_0}^{x_n} y dx = nh\left[y_0 + \frac{n}{2}\Delta y_0 + \frac{n(2n-3)}{12}\Delta^2 y_0 + \frac{n(n-2)^2}{24}\Delta^3 y_0 + \cdots \right] \quad (4)$$

This is a general quadrature formula and is known as Newton-Cotes quadrature formula. Trapezoidal rule, Simpson's one-third, and Simpson's three-eighth rules can be respectively derived by putting n = 1, 2, and 3 in the formula (4)

0.2.2 Trapezoidal Rule

Setting $n = 1$ in (4) and taking the curve through (x_0, y_0) and (x_1, y_1) as a polynomial of degree one so that differences of an order higher than one vanish, we get

$$\int_{x_0}^{x_0+h} ydx = h\left(y_0 + \frac{1}{2}\Delta y_0\right) = \frac{h}{2}\Big[2y_0 + (y_1 - y_0)\Big] = \frac{h}{2}(y_0 + y_1).$$

Similarly, for the next sub-interval $(x_0 + h, x_0 + 2h)$, we get

$$\int_{x_0+h}^{x_0+2h} ydx = \frac{h}{2}(y_1 + y_2), \cdots, \int_{x_0+(n-1)h}^{x_0+nh} ydx = \frac{h}{2}(y_{n-1} + y_n)$$

Adding the above integrals, we get

$$\int_{x_0}^{x_0+nh} ydx = \frac{h}{2}\Big[y_0 + 2\big(y_1 + y_2 + \cdots + y_{n-1}\big) + y_n\Big] \tag{5}$$

which is the Trapezoidal rule.

0.2.3 Simpson's 1/3 - Rule

Putting $n = 2$ in formula (4) and taking the curve through (x_0, y_0), (x_1, y_1) and (x_2, y_2) as a polynomial of degree two so that differences of order higher than two vanish, we get

$$\int_{x_0}^{x_0+2h} ydx = 2h\left(y_0 + \Delta y_0 + \frac{1}{6}\Delta^2 y_0\right)$$

$$= \frac{2h}{6}\Big[6y_0 + 6\big(y_1 - y_0\big) + \big(y_2 - 2y_1 + y_0\big)\Big]$$

$$= \frac{h}{3}\Big(y_0 + 4y_1 + y_2\Big)$$

Similarly,

$$\int_{x_0+2h}^{x_0+4h} ydx = \frac{h}{3}\Big(y_2 + 4y_3 + y_4\Big), \cdots,$$

$$\int_{x_0+(n-2)h}^{x_0+nh} f(x)dx = \frac{h}{3}\Big(y_{n-2} + 4y_{n-1} + y_n\Big).$$

Adding the above integrals, we get

$$\int_{x_0}^{x_0+nh} ydx = \frac{h}{3}\Big[y_0 + 4(y_1 + y_3 + \cdots + y_{n-1}) + 2(y_2 + y_4 + \cdots + y_{n-2}) + y_n\Big] \quad (6)$$

which is Simpson's one-third rule.
Note: Simpson's one-third rule rule requires the whole range (the given interval) must be divided into even number of equal subintervals.

0.2.4 Simpson's 3/8 - Rule

Similarly, setting $n = 3$ **in formula** (4) **and taking the curve through** (x_0, y_0), (x_1, y_1), (x_2, y_2) **and** (x_3, y_3) **as a polynomial of degree three so that differences of order higher than three vanish, we get**

$$\int_{x_0}^{x_0+3h} ydx = 3h\Big(y_0 + \frac{3}{2}\Delta y_0 + \frac{3}{4}\Delta^2 y_0 + \frac{1}{8}\Delta^3 y_0\Big)$$

$$= \frac{3h}{8}\Big[8y_0 + 12\Big(y_1 - y_0\Big) + 6\Big(y_2 - 2y_1 + y_0\Big) + \Big(y_3 - 3y_2 + 3y_1 - y_0\Big)\Big]$$

$$= \frac{3h}{8}\Big[y_0 + 3y_1 + 3y_2 + y_3\Big].$$

Similarly,

$$\int_{x_0+3h}^{x_0+6h} ydx = \frac{3h}{8}\Big[y_3 + 3y_4 + 3y_5 + y_6\Big], \cdots$$

$$\int_{x_0+(n-3)h}^{x_0+6h} ydx = \frac{3h}{8}\Big[y_{n-3} + 3y_{n-2} + 3y_{n-1} + y_n\Big].$$

Adding the above integrals, we get

$$\int_{x_0}^{x_0+nh} ydx = \frac{3h}{8}\Big[y_0 + 3(y_1 + y_2 + y_4 + y_5 + \cdots + y_{n-2} + y_{n-1}) + 2(y_3 + y_6 + \cdots + y_{n-3}) + y_n\Big] \quad (7)$$

which is known as Simpson's three-eighth rule.
Note: This rule also requires that the number of subintervals should be taken as multiples of 3.

Example 5. *Evaluate*

$$\int_0^1 \frac{1}{1+x}dx$$

using trapezoidal and Simpson's rules with $h = 0.5, 0.25,$ *and* 0.125, *correct to three decimal places.*

Solution: (i) $h = 0.5:$ **The values of** x **and** y **are tabulated below (Table** 2**)**

(a) Trapezoidal rule gives

$$I = \frac{1}{4}\Big[1.0000 + 2(0.6667) + 0.5000\Big] = 0.70835.$$

(b) Simpson's $\frac{1}{3}$ **rule gives**

$$I = \frac{1}{6}\Big[1.0000 + 4(0.6667) + 0.5000\Big] = 0.6945.$$

(ii) $h = 0.25:$ **The tabulated values of** x **and** y **are given below (Table** 3**)**

(a) Trapezoidal rule gives

$$I = \frac{1}{8}\Big[1.0000 + 2(0.8000 + 0.6667 + 0.5714) + 0.5000\Big] = 0.6970.$$

(b) Simpson's $\frac{1}{3}$ **rule gives**

$$I = \frac{1}{12}\Big[1.0000 + 4(0.8000 + 0.5714) + 2(0.6667) + 0.5000\Big] = 0.6932.$$

(iii) $h = 0.125:$ **The tabulated values of** x **and** y **are given below (Table** 4**)**

(a) Trapezoidal rule gives

$$I = \frac{1}{16}\Big[1.0{+}2(0.8889{+}0.80{+}0.7273{+}0.6667{+}0.6154{+}0.5714{+}0.5333){+}0.50\Big] = 0.6941.$$

(b) Simpson's $\frac{1}{3}$ **rule gives**

$$I = \frac{1}{24}\Big[1.0{+}4(0.8889{+}0.7273{+}0.6154{+}0.5333){+}2(0.80{+}0.6667{+}0.5714){+}0.5\Big] = 0.6932.$$

Hence the value of I **may be taken to be** 0.693, **correct to three decimal places.**
The exact value of I **is** $\log_e 2$, **which is** $0.693147\cdots$.
This example demonstrates that, in general, Simpson's rule yields more accurate results than the trapezoidal rule.

EXERCISE 3

(a) **Evaluate**

$$(i)\quad \int_0^{\pi} x\sin(x)dx \quad and \quad (ii)\quad \int_{-2}^{2} \frac{x}{5+2x}dx$$

using the trapezoidal rule with five ordinates.

(b) **Using Simpson's $\frac{1}{3}$ rule with $h = 1$, evaluate the integral**

$$I = \int_3^7 x^2 \log x dx.$$

(c) **Estimate the value of integral**

$$I = \int_0^{\frac{1}{2}} \frac{dx}{\sqrt{x}\sqrt{1-x}}$$

using the trapezoidal rule. What is its exact value?

(d) **Evaluate**

$$\int_0^2 \frac{dx}{x^3+x+1}$$

by using Simpson's $\frac{1}{3}$ rule with $h = 0.25$.

(e) **Suppose that $f(0) = 1, f(0.5) = 2.5, f(1) = 2,$ *and* $f(0.25) = f(0.75) = \alpha$. Find α if the Trapezoidal rule with $n = 4$ gives the value 1.75 for $\int_0^1 f(x)dx$.**

Table 1: Table of Results

n	a_n	b_n	r_n	$f(r_n)$
1	1.0	2.0	1.5	2.375
2	1.0	1.5	1.25	−1.79687
3	1.25	1.5	1.375	0.16211
4	1.25	1.375	1.3125	−0.84839
5	1.3125	1.375	1.34375	−0.35098
6	1.34375	1.375	1.359375	−0.09641
7	1.359375	1.375	1.3671875	0.03236
8	1.359375	1.3671875	1.36328125	−0.03215
9	1.36328125	1.3671875	1.365234375	0.000072
10	1.36328125	1.365234375	1.364257813	−0.01605
11	1.364257813	1.365234375	1.364746094	−0.00799
12	1.364746094	1.365234375	1.364990235	−0.00396
13	1.364746094	1.365234375	1.365112305	−0.00194

Table 2: Table of Values

x	y
0.0	1.0000
0.5	0.6667
1.0	0.5000

Table 3: Table of Values

x	y
0.00	1.0000
0.25	0.8000
0.50	0.6667
0.75	0.5714
1.00	0.5000

Table 4: Table of Values

x	y
0.000	1.0000
0.125	0.8889
0.250	0.8000
0.375	0.7273
0.500	0.6667
0.625	0.6154
0.750	0.5714
0.875	0.5333
1.000	0.5000

OBAFEMI AWOLOWO UNIVERSITY, ILE-IFE, NIGERIA

Department of Mathematics

COURSE : MTH 201 -MATHEMATICAL METHODS I - Differential Equations I.
COURSE UNITS: 4
COURSE LECTURERS:
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Differential Equations I

In this section, we shall be considering the following:

- Introduction to ordinary differential equations (ode);
- classifications (order, degree, linear/non-linear etc.).
- Separable, Homogeneous and Exact ode.

Introduction to ordinary differential equations (ode)

To start with, what is a differential equation?

A differential equation is any relation that involve a dependent variable say y that is depending on an independent variable say x and derivatives of the dependent variable with respect to the independent varaible.

$$f(x; y(x), y'(x), y''(x), \ldots) = 0 \qquad (1)$$

equivalently, equation of the following forms are called differential equations

$$\frac{dy}{dx} + 3y = 0 \tag{2}$$

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 10y = \sin x \tag{3}$$

$$\frac{d^xy}{dx^3} - 4\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = 4 \tag{4}$$

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 2u(x, y) \tag{5}$$

$$u_{xx} + u_{yy} = 0 \tag{6}$$

Classification of Differential Equations

Differential equation can be classified as follows:

- By Type
- By Order
- By Nature
- By Degree

By Type: This is the classification based on the dependent and independent variables present in the equation.

There are five main types of differential equations currently of interest to scientist and engineers now this are

- Ordinary Differential Equation - ODE
- Partial Differential Equation - PDE
- Delay Differential Equation - DDE
- Stochastic Differential Equation - SDE
- Franctional Order Differential Equation - FDE

Ordinary Differential Equation - ODE: This is an equation where the dependent variable is depending only on one independent variable.

$$f(x; y(x), y'(x), y''(x), \ldots) = 0$$

In the equation above, the dependent variable is y and x is the independent variable.
Equations (2)-(4) above are examples od ODE.

Partial Differential Equation - ODE: This is an equation where the dependent variable is depending on more than one independent variables.

$$f(x, y; u(x, y), u_x, u_y, u_{xy}, \ldots) = 0$$

In the above equation, u is the dependent variable while variables x and y are the independent variables. u_x, u_y and u_{xy} are the derivatives of u with respect to x, y and second derivative of u in the order of the subscripts indicated x first then y. Equation (5) above is an example of a PDE.

Delay Differential Equation - ODE: A delay differential equation is a differential equation where the time derivatives at the current time depend on the solution and possibly its derivatives at previous times: Instead of a simple initial condition, an initial history function needs to be specified. The quantities and are called the delays or time lags.

Stochastic Differential Equation - ODE: A stochastic differential equation (SDE) is a differential equation in which one or more of the terms is a stochastic process, resulting in a solution which is also a stochastic process. SDEs are used to model various phenomena such as unstable stock prices or physical systems subject to thermal fluctuations

Fractional Differential Equation - ODE: Fractional order differential equations are generalized and noninteger order differential equations, which can be obtained in time and space with a power law memory kernel of the nonlocal relationships; they provide a powerful tool to describing the memory of different substances and the nature of the inheritance.

Classification by Order: Differential equations can be calsssified by order. So what is the order of a differential equation?

The **order** of a differential equation is the highest differential coefficient present in the equation. e.g.Equation (2) is a first order ordinary differential equation while equations (3) and (4) are second order and third order ordinary differential equations respectively. Equations (5) and (6) are respectively first order and second order PDE.

Classification by Nature: Here we want to classify based on either the equation is linear, nonlinear, homogeneuous and non homogeneouous.

Linear Equation: A differential equation is said to be linear if the dependent variable (or variables) and their derivatives appear linearly, that is only as first powers, they are not multiplied together, and no other functions of the dependent variables appear. i.e.

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = b(x) \tag{7}$$

The functions $a_0(x) \ldots a_n(x)$ are called coefficients. Any differential equation that violates the above definition is called a nonlinear differential equation.

Homogeneous Equation: Consider the equation (7), if $b(x) = 0$, the linear equation is said to be or called Homogeneous otherwise **nonhomogeneous**.

Exercise 1. Determine if each of the following equations is linear or nonlinear. If linear, determine if its is homogeneous or nonhomogeneous.

1. $\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} - 1 = 0$
2. $\left(\dfrac{dy}{dx}\right)^3 = x + y$
3. $y\left(\dfrac{dy}{dx}\right) = x$
4. $x^3\left(\dfrac{d^2y}{dx^2}\right) - 3x^2\dfrac{dy}{dx} = y$
5. $\dfrac{dy}{dx} = e^x + e^y$
6. $x^2\left(\dfrac{d^2y}{dx^2} + x\right) = xy - y$

Exercise 2: Classify the following equations. Are they ODE or PDE? What is the order? Is it linear or nonlinear, and if it is linear, is it homogeneous or nonhomogeneous?

1. $\dfrac{\partial^2 u}{\partial x^2} + 3\dfrac{\partial^2 u}{\partial y^2} = \sin x$
2. $\dfrac{dy}{dx} + \cos xy = x^2 + x + 1$
3. $\dfrac{d^5 y}{dx^5} = 3y$
4. $\dfrac{d^2 y}{dx^2} + 8\dfrac{dy}{dx} = 1$
5. $\dfrac{\partial u}{\partial x} = \dfrac{\partial^2 u}{\partial y^2} + u^2$

To conclude this class, let us describe classification based on the degree of the equation.
What is the degree of a differential equation? **Degree of Differential Equation:** The degree of a differential equation is defined as the power to which the highest order derivative is raised. E.g.

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 1 = 0$$

is a Second order ODE with degree one while

$$\left(\frac{dy}{dx}\right)^3 = x + y$$

is a first order ODE with degree 3

In our next class, we shall define solution of differential equation and narrow ourselves to considering First order ODE, types as well as method of obtaining solutions based on the type of the equation.

Thank You

OBAFEMI AWOLOWO UNIVERSITY, ILE-IFE, NIGERIA

Department of Mathematics

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COURSE UNITS: 4
COURSE LECTURERS:
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In our last class, we introduced ourselves to differential equations by considering its definition, classifications based on type, order, nature and degree.
Today we want to define solution of differential equations and look into details the first order ordinary differential equations and method of solving some types of first order ODE.

Solution of a differential Equation:
Consider the diferential equation

$$f(x, y(x), y'(x), \dots, y^n(x)) = 0 \quad (1)$$

Any function $\phi(x)$ that will satisfy the equation (1) is called the solution of the differential equation.
i.e

$$f(x, \phi(x), \phi'(x), \dots, \phi^n(x)) = 0$$

For example, the equation

$$y' - y = 0$$

is only satisfied by the function $\phi(x) = e^x$, i.e.

$$\phi'(x) - \phi(x) = 0$$

$$e^x - e^x = 0$$

and $\phi(x) = e^{-x}$ will satisfy the equation

$$y' + y = 0$$

i.e.

$$\phi'(x) + \phi(x) = 0$$

$$-e^x + e^x = 0$$

Consider the second order differential equation

$$y'' + y = 0 \tag{2}$$

The functions $\phi(x) = \sin x$ and $\phi(x) = \cos x$ will both satisfy the equation (2).
For $\phi(x) = \sin x$, we have
$\phi'(x) = \cos x$ and $\phi''(x) = -\sin(x)$.
Substituting thesee into the differential equation (2) we have

$$\sin x + (-\sin x) = 0$$

Clearly, it can be seen that for the first order equation there is only one function that will satisfy the equation but for the second order equation, two different functions will satisfy it. Indeed it can be deduced that the function and its first derivative will satisfy the second order equation.

From the examples above it is clear that the equation (1) will have n solutions which can be said to be the function $\phi(x)$ and its first $(n-1)$ derivatives, i.e. $\phi x, \phi'(x), \phi''(x) \ldots, \phi^{n-1}(x)$ which results in n solutions.

First Order ODE

We now want to consider the first order Ordinary differential equations and some methods of solving them.
The following class of ifrst order ODE will be consider in my section of the course while the next lecturer will take further class and methods of solving them.

1. Separable Equation
2. Homegeneous Equation
3. Exact Equation

Separable Equation

Consider the general first order ordinary differential equation

$$y' = f(x, y) \tag{3}$$

An equation is saidd to be separable if $f(x, y)$ can be decomposed into product or quotient of the functions of x and y i.e.

$$f(x, y) = h(x)g(y)$$

or

$$f(x, y) = \frac{h(x)}{g(y)}$$

such that equation (3) becomes

$$y' = h(x)g(y)$$

or

$$y' = \frac{h(x)}{g(y)}$$

Recall that $y' = \dfrac{dy}{dx}$, re-writing the above equations as

$$\frac{dy}{dx} = h(x)g(y) \tag{4}$$

or

$$\frac{dy}{dx} = \frac{h(x)}{g(y)} \tag{5}$$

It could be seen clearly that variables can be separated to sides of the equality sign hence the name separatble equations. In a more simplified form, equations (4) and (5) can be put in this form

$$\frac{dy}{g(y)} = h(x)dx \tag{6}$$

or

$$g(y)dy = h(x)dx \tag{7}$$

To solve simply means to integrate both sides of (6) and (7)

$$\int \frac{dy}{g(y)} = \int h(x)dx \tag{8}$$

or

$$\int g(y)dy = \int h(x)dx \tag{9}$$

Examples: Solve the following first order differential equations

1. $y' = 3x^2y$
2. $y' = \dfrac{(x^2 - 4x + 2)}{3y}$
3. $y' = e^{-y}(x^3 + 2)$
4. $y' = e^{(y-x)} \sec y(1 + x^2)$

Solutions:
Consider
$$y' = 3x^2y$$
It is clear that the function $f(x, y) = 3x^2y$ is separable since it is a product of functions $h(x) = 3x^2$ and $g(y) = y$ hence we can re-write the equation as
$$\frac{dy}{g(y)} = h(x)dx$$
which yields
$$\frac{dy}{y} = 3x^2dx$$
Integrating both sides we have

$$\int \frac{dy}{y} = \int 3x^2 dx$$

which gives

$$\ln y = x^3 + C$$

and

$$y(x) = e^{x^3+C}$$

$$y(x) = Ke^{x^3}$$

C and K are arbtrary constant.

Consider

$$y' = \frac{x^2 - 4x + 2}{3y}$$

Also it this clearly separable and so we have

$$\int (3y)dy = \int (x^2 - 4x + 2)dx$$

which yields

$$\frac{3}{2}y^2 = \frac{1}{3}x^3 - 2x^2 + 2x + C$$

which reduces to

$$y(x) = \pm\sqrt{\frac{2}{3}(\frac{1}{3}x^3 - 2x^2 + 2x + C)}$$

Consider

$$y' = e^{-y}(2x - 4)$$

This is also separable,

$$\int (e^{y})dy = \int (2x - 4)dx$$

which on solving we have

$$e^{y} = x^2 - 4x + C$$

.Thus

$$y(x) = \ln\left(x^2 - 4x + C\right)$$

Consider lastly,

$$y' = e^{(y-x)} \sec y(1 + x^2)$$

This is also separable and can be put in the form

$$y' = e^y \sec y \quad e^{-x}(1 + x^2)$$

which yields

$$\int (e^y \cos y)dy = \int e^{-x}(1 + x^2)dx$$

Using integration by parts we have

$$\frac{1}{2}(e^y(\cos y - \sin y)) = -e^{-x}[(1 + x^2) + 2x + 2] + C$$

which can be written as

$$e^y(\cos y - \sin y) = -2e^{-x}[(1 + x^2) + 2x + 2] + C$$

Exercises: Solve

1. $y' = \dfrac{x+1}{y^4+1}$
2. $(t+1)dt - \frac{1}{y^2}dy = 0$
3. $y' = \dfrac{xe^x}{2y}$
4. $(x^2+1)dy - \frac{1}{y}dy = 0$
5. $y' = \dfrac{x^2y - y}{y+1}$

In our next class, we shall be considering the Homogenous equation and how to solve them.
When is a differential equation of first order said to be homogenous?

Thank You

OBAFEMI AWOLOWO UNIVERSITY, ILE-IFE, NIGERIA

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Differential Equations I.
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In our last class, we considered what is a solution of a differential equation and went on to consider the first order ordinary differential equation. Some classes of first order equations were identified and we treated the class of separable equation and method of solving them. In this class, we want to consider a class reffered to as Homogenous equation and method of solving them.

Homogenous Equation

Consider the diferential equation

$$y' = f(x, y) \tag{1}$$

The above is called the standard form of first order differential equation. The equivalent differential form of equation (2) is given as

$$M(x, y)dx + N(x, y)dy = 0 \tag{2}$$

A differential equation in standard form (2) is homogeneous if

$$f(tx, ty) = t^n f(x, y)$$

Here the degree of homogenousity is n

For example, determine if the following equations are homogenous

1. $y' = \dfrac{y+x}{x}$
2. $y' = \dfrac{y^2}{x}$
3. $y' = \dfrac{2xye^{x/y}}{x^2 + y^2 \sin\left(\frac{x}{y}\right)}$
4. $y' = \dfrac{x^2y}{x^2y + y^3}$
5. $y' = \dfrac{-xy^2}{x^2y + y^2}$

To solve homogeneous differential equation, the transformation $y = vx \quad or \quad v = \frac{y}{x}$ is made to reduce the problem to a separable equation which by now we can solve without as learnt from the last class
From the transformation

$$y = vx \quad or \quad v = \frac{y}{x},$$

we have

$$y' = v + xv'$$

i.e.

$$\frac{dy}{dx} = v + \frac{dv}{dx}$$

Using the transformation the (2) becomes

$$v + \frac{dv}{dx} = f(v) \tag{3}$$

simpilifying (3) we have

$$\frac{dv}{dx} = f(v) - v = g(v) \tag{4}$$

which reduces to

$$\frac{dv}{dx} = g(v)$$

which is a saparable equation problem.

Solving this yields

$$\frac{dv}{g(v)} = dx \tag{5}$$

integrating both sides

$$\int \frac{dv}{g(v)} = \int dx \tag{6}$$

After solving, we replace the function $v(x)$ with $\frac{y}{x}$ and the solution to the original equation is obtained.

Examples: Solve the following Homogeneous differential equation

1. $y' = \dfrac{y+x}{x}$
2. $y' = \dfrac{2y^4 + x^4}{xy^3}$

Solution: Since the differential equation is an homogeneous equation, divide both the numerator and denominator on the LHS by x. We have

$$y' = \frac{y}{x} + 1$$

using the transformation $v = \frac{y}{x}$ and $y' = v + xv'$, we have

$$v + xv' = v + 1.$$

On further simplification we have the new equation as

$$v' = \frac{1}{x}$$

this yields

$$\int dv = \int \frac{dx}{x}$$

and

$$v(x) = \ln x + C$$

using $v = \frac{y}{x}$, we have

$$\frac{y}{x} = \ln x + C$$

this gives the solution as

$$y(x) = x\left(\ln x + C\right)$$

For the problem

$$y' = \frac{2y^4 + x^4}{xy^3}$$

The equation is homogeneous of degree 4, hence we divide both th numerator and denominator of the RHS by x^4

$$y' = \frac{2(\frac{y}{x})^4 + 1}{(\frac{y}{x})^3}$$

using the transformation $v = \frac{y}{x}$ we have that

$$v + x\frac{dv}{dx} = \frac{2v^4 + 1}{v^3}$$

simplifying the above gives

$$x\frac{dv}{dx} = \frac{v^4+1}{v^3}$$

Inegrating both sides

$$\int \frac{v^3}{v^4+1}dv = \int \frac{dx}{x}$$

this gives

$$\frac{1}{4}\ln\left(v^4+1\right) = \ln x + C$$

which is

$$\ln\left(v^4+1\right) = 4(\ln x + C) = 4(\ln(Cx))$$

hence

$$v^4+1 = (Cx)^4$$

$$v(x) = \left(\sqrt{(Cx)^4-1}\right)^{\frac{1}{4}}$$

This finally yields

$$y(x) = x\left(\sqrt{(Cx)^4 - 1}\right)^{\frac{1}{4}}$$

or

$$y^4(x) = C_1x^8 - x^4$$

where $C_1 = C^4$

Exercises: Solve the following first order Homogeneous equation.

1. $y' = \dfrac{2xy}{y^2 - x^2}$
2. $y' = \dfrac{x^2 + y^2}{2xy}$
3. $y' = \dfrac{x^2 + 2y^2}{xy}$
4. $y' = \dfrac{x^4 + 3x^2y^2 + y^4}{x^3y}$

In our next class, we shall consider Exact differential equations method of solving them and how make equations that are not exact to be exact.

Thank You

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COURSE : MTH 201 -MATHEMATICAL METHODS I -
Differential Equations I.
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In our last class, we considered the homogeneous differential equations and method of solving them. Today, we want to consider Exact differential equation and how to solve them.

Exact Equation

Recall the standard form of first order ordinary diferential equation

$$y' = f(x, y) \tag{1}$$

with equivalent differential form given as

$$M(x, y)dx + N(x, y)dy = 0 \tag{2}$$

(2) is said to be exact is if there exists a function $h(x, y)$ such that

$$dh(x, y) = M(x, y)dx + N(x, y)dy \tag{3}$$

Test for exactness: If $M(x, y)$ and $N(x, y)$ are continuous functions and have continuous first partial derivatives on some rectangle of the xy-plane, then (3) is exact if and only if

$$\frac{\partial M(x,y)}{\partial y} = \frac{\partial N(x,y)}{\partial x} \tag{4}$$

How do we solve exact equation problem?

Method of Solution:

To solve the exact equation (3), we need to solve the equations

$$\frac{\partial h(x,y)}{\partial x} = M(x,y) \tag{5}$$

and

$$\frac{\partial h(x,y)}{\partial y} = N(x,y) \tag{6}$$

for $h(x,y)$.

The solution for the equation (2) is given implicitly as

$$h(x,y) = K \tag{7}$$

where K is constant

Equation (16) is immediate from Eqs. (2) and (3). If (3) is substituted into (2), we obtain

$$dh(x, y(x)) = 0.$$

Integrating this equation (note that we can write 0 as $0dx$), we have

$$dh(x, y(x)) = 0dx,$$

which, in turn, implies (16).

Determine if the following equations are exact or not, if exact find the solution

1. $ydx - xdy = 0$
2. $2xydx + (1 + x^2)dy = 0$
3. $(x + \sin y)dx + (x \cos y - 2y)dy = 0$
4. $y^2 dt + (2yt + 1)dy = 0$

Solutions

For

$$ydx - xdy = 0$$

$$M(x,y) = y \quad \text{and} \quad N(x,y) = x$$

$$\frac{\partial M(x,y)}{\partial y} = 1 \quad \text{and} \quad \frac{\partial N(x,y)}{\partial x} = -1,$$

since

$$\frac{\partial N(x,y)}{\partial x} \neq \frac{\partial M(x,y)}{\partial y}$$

the equation is not Exact

For

$$2xydx + (1 + x^2)dy = 0$$

$$M(x, y) = 2xy \quad \text{and} \quad N(x, y) = (1 + x^2)$$

$$\frac{\partial M(x, y)}{\partial y} = 2x \quad \text{and} \quad \frac{\partial N(x, y)}{\partial x} = 2x,$$

since

$$\frac{\partial N(x, y)}{\partial x} = \frac{\partial M(x, y)}{\partial y}$$

the equation is Exact

For

$$(x + \sin y)dx + (x\cos y - 2y)dy = 0$$

$$M(x, y) = (x + \sin y) \quad \text{and} \quad N(x, y) = (x\cos y - 2y)$$

$$\frac{\partial M(x, y)}{\partial y} = \cos y \quad \text{and} \quad \frac{\partial N(x, y)}{\partial x} = \cos y,$$

since

$$\frac{\partial N(x, y)}{\partial x} = \frac{\partial M(x, y)}{\partial y}$$

the equation is Exact

For

$$y^2 dt + (2yt + 1)dy = 0$$

$$M(t, y) = y^2 \text{ and } N(t, y) = (2yt + 1)$$

$$\frac{\partial M(x, y)}{\partial y} = 2y \text{ and } \frac{\partial N(x, y)}{\partial x} = 2y,$$

since

$$\frac{\partial N(x, y)}{\partial x} = \frac{\partial M(x, y)}{\partial y}$$

the equation is Exact

To solve the exact equation

$$2xydx + (1 + x^2)dy = 0,$$

we need to solve the equations

$$\frac{\partial h(x,y)}{\partial x} = M(x,y) \tag{8}$$

and

$$\frac{\partial h(x,y)}{\partial x} = N(x,y) \tag{9}$$

which is

$$\frac{\partial h(x,y)}{\partial x} = 2xy \tag{10}$$

Integrating yields

$$h(x,y) = x^2y + f(y) \tag{11}$$

where $f(y)$ is a constant of integration.

differentiating (11) with respect to y we have

$$\frac{\partial h(x,y)}{\partial y} = x^2 + f'(y) \tag{12}$$

comapring (12) and (9) we have

$$N(x,y) = x^2 + f'(y) \tag{13}$$

which is

$$(1 + x^2) = x^2 + f'(y) \tag{14}$$

simplifying yields

$$f'(y) = 1$$

hence

$$h(x,y) = x^2y + y + C \tag{15}$$

Substitute (15) into (11) we have

$$h(x, y) = x^2y + y + C$$

which can be given implicitly as

$$x^2y + y = C \qquad (16)$$

where C is constant.

Students are encouraged to follow the procedures to solve other exact equations identified above.

When an equation is not exact, can it made exact? **YES** by seeking an appropriate factor called an **Integrating factor**.The original equation will be multiplied by the integrating factor then the neww equation becomess exact and the solution can be found through the process of solving exact equations

A function $I(x, y)$ is an **integrating factor** for (2) if the equation

$$I(x, y)[M(x, y)dx + N(x, y)dy] = 0$$

is exact. **Integrating Factor I(x,y)**

1. If $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \equiv g(x)$ a function of x alone,

$$I(x, y) = e^{\int g(x)dx}$$

2. If $\frac{1}{M}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \equiv h(y)$ a function of y alone,

$$I(x, y) = e^{-\int h(y)dy}$$

1. If $M = yf(xy)$ and $N = xg(xy)$ then

$$I(x,y) = \frac{1}{xM - yN}$$

Example

For the problem

$$ydx - xdy = 0 \tag{17}$$

$$M(x,y) = y \quad \text{and} \quad N(x,y) = x$$

clearly, this equation is not exact.

$\dfrac{1}{N}\left(\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}\right) \equiv g(x)$ a function of x alone, i.e

$-\dfrac{1}{2}(1+1) \equiv -\dfrac{2}{x}$ a function of x alone,

$$I(x,y) = e^{\int -\frac{2}{x}dx}$$

$$I(x,y) = -\frac{1}{x^2}$$

Multiply (17) by $-\frac{1}{x^2}$, we have the new equation

$$-\frac{y}{x^2}dx + \frac{1}{x}dy = 0 \qquad (18)$$

This new equation is exact and hence can be solved using the procedure of solving exact equation.

Students are encouraged to practice the exercises below to enrich their understanding

Solve

1. $(y^2 - y)dx + xdy = 0.$
2. $(-2xy + x)dx + dy = 0$
3. $y^2 dx + xydy = 0$
4. $ydx + 3xdy = 0$

Thank You

MTH 201- MATHEMATICAL METHODS I

DIFFERENTIAL EQUATIONS LECTURE 001

Inaugural Interactive Discussion/Assignment 07-11-2022

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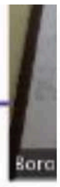
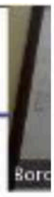
Concept Digest: ONE



- Could a *Relation* $\mathcal{R}$ be said to be
  - A process of taking elements from set **X** to obtain some elements belonging to another set **Y**?
  - The rule of acting on elements from one set to obtain elements of another set: $\mathcal{R} : \mathbf{X} \longrightarrow \mathbf{Y}$?
- Could *Domain* be described as *a set on which* (or a *region where*) it makes sense to talk of a *Relation*?
- And by extension, given appropriate condition, a *Function*?
- Justify.
- In discussing *Relation* vis-a-vis *Domain* and *Co-domain*, which types of *relation* exist?
- When is a *Relation* said to be (i.) *One-One*, (ii.) *Many-One* and (iii.) *One-Many*?

Concept Digest: ONE ContD

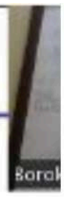


- Which of these relations above endows a *Function* or a *Map* or *Mapping*?
- When is *Mapping* said to be (i.) *Injective*, (ii.) *Surjective* and (iii.) *Bijective*?
- When in each of these can we talk of *Inverse Mapping* or *Inverse Function*?
- Why will *Many-One Relation* constitute a *Function*?
- Why will *One-Many Relation* not constitute a *Function*?
- What is the import of *Surjectivity* in respect of mapping?

Concept Digest: TWO

- Distinguish between the concepts: *limit*, *continuity* and *derivative*, for a given function $f(x)$ of an independent variable x on an interval of a *real line*.
- What do we understand by the term *derivative*, in general?
- Compare the concept in (a.) with that conveyed by the term *integral*.
- What do we make of the terms *Antiderivative*?
- Distinguish between the terms *derivative* and *differential*.
- What is the common feature in the terms *differentiation* and *integration*? How are they different?
- (i) Give 5 examples of functions and their derivatives. (ii) Find from *first principle* the derivative of the functions: $3x^2$, $\cos x$, $\sin 2x$ $\tan x$.

Concept Digest: THREE

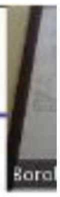


- How do *differential equations* arise? How do they relate to physical processes in: (i) Engineering? (ii) Economics, (iii) Ecology, (iv) Other social phenomena?
- Distinguish between the *Order* and *Degree* of a differential equation.
- What do we understand by:
  - Homogeneous function,
  - Homogeneous equation,
  - Periodic function,
  - Odd function and Even function.

Concept Digest: FOUR

- Differentiate the function:
  (i) $3x^2 - 2x$, (ii) $x\cos 2x$, (iii) $x\sin x + \tan x$, xe^{2x} and (iv) $\ln 2x$.
- Find the indefinite integral of the following functions with respect to the independent variable x:
  (i) $4x^3 + e^2x$, (ii) $\frac{2x}{3+x^2} + \tan x$ and (iii) $\sin 2x + xe^{x^2}$.
- What is an *isocline*, with respect to an ode?
- Give the *Binomial Expansion* of each of the functions $(1-x)^n$ and $(1+x)^n$ when:
  (i) $n \in \mathcal{N}$, (ii) $n \in \mathcal{Z}$ and (iii) $n \in \mathcal{R}$.

Concept Digest: FIVE



- What is the *Maclaurin's expansion* of a function?
- Give the *1st four terms* of the Maclaurin's expansion for the functions:
  (i) $\cos x$, (ii) $\sin x$, (iii) $\tan x$, (iv) e^x, (v) e^{-x}, (vi) $\ln(1+x)$, (vii) $\ln(1-x)$ and (viii) $\ln\frac{1}{1+x}$.
- Give the *approximation* for each of those functions, with the *1st two terms* of the expansion, as x tends to.
- (i) Sketch the curves of each function as $x \longrightarrow 0$. (ii) Compare, where possible, those curves to the curves of $x, x^2, -x^2, x^3$ on the same graph.

Module and Outline

- **Aspect of Solution of 1st Order ODE** (Solutions of first order ordinary differential equations (ODE))
- Linear differential equations; Bernoulli's equations and Riccati's equations.
- (**Pre-requisite:** Introductory Basics of ODE)

Textbooks

1. Mathematical Methods I: *A self-guide to the rudiments.* Department of Mathematics, OAU, Ife.
2. Advanced Engineering Mathematics, Erwin Kreyszig.
3. Mathematical Methods. K. F. Riley, M. P. Hobson and S. J. Bence.
4. Collection of Problems on Differential Equations, A. F. Fillipov
5. Pure Mathematics for Advanced Level B. D. Bunday and H. Mulholland.
6. Any book(s) on: Advanced Mathematics, Mathematical methods, Differential Equations, Mechanics, Modelling, ...

MTH 201- MATHEMATICAL METHODS I

DIFFERENTIAL EQUATIONS LECTURE 002

Introduction to Differential Equations

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Differential Equations

- An equation that contains derivative(s). OR
- An equation containing derivative(s) of unknown function(s). OR
- An equation that relates two or more variables in terms of derivative(s) or differentials. OR
- An equation that relates the derivative of a dependent variable to an independent variable.

Differential Equations ContD

- We generally write

$$\Phi(\vec{x}, y(\vec{x}), y'(\vec{x}), y''(\vec{x}), \ldots, y^m(\vec{x})) = 0;$$

$$\vec{x} = (x_1, x_2, \ldots, x_n),\ \vec{x} \in \mathcal{R}^n,\ m, n \in \mathcal{N}, \tag{1.1}$$

  and say we have a *Partial Differential Equation* (PDE).

- But in the case when $n = 1$, we write

$$\Phi(x, y(x), y'(x), y''(x), \ldots, y^m(x)) = 0,$$

$$x \in \mathcal{R}, \quad m \in \mathcal{N} \tag{1.2}$$

  and obtain an *Ordinary Differential Equation* (ODE).

- In either (1.1) or (1.2), m signifies the *order* of the differential equation (DE).

Differential Equation ContD

- In the one-dimensional case, (1.2), we obtain in either the *explicit form* or, as above, the *implicit form*:
- $m = 0 \quad \longleftrightarrow \quad$ Algebraic/Polynomial equation;

$$y = P(x) \;\; or \;\; P(x, y) = 0.$$

- $m = 1 \quad \longleftrightarrow \quad$ 1st Order ODE.; $\quad y' = f(x, y)$
  i.e. $\Phi(x, y, y'(x)) = 0$.
- $m = 2 \quad \longleftrightarrow \quad$ 2nd Order ODE; $\quad y'' = f(x, y, y')$
  i.e. $\Phi(x, y, y'(x), y''(x)) = 0. \vdots$
- $m = k \quad \longleftrightarrow \quad k^{th}$ Order ODE; $\quad y^{(k)} = f(x, y' \ldots y^{(k-1)})$
  i.e. $\Phi(x, y, y'(x) \ldots y^{k}(x)) = 0$.

Solutions of ODE: General Solution

- **solution** (also referred to as *integral* or *integral curve*), $y = \varphi(x)$, of an ode is that function $\varphi(x)$ which reduces the ode to an identity when substituted into it.
- i.e. that function $\varphi(x)$ that satisfies a given ode identically.

- **General solution**
  The <u>*General solution*</u> (i.e. totality of all special/particular solutions) is that function with m-number of constants as the order of the ode:

$$y = \varphi(x, c_1, c_2, \ldots c_m),$$

where $c_1, c_2, \ldots c_m$ are constants found from specified *boundary conditions* or *initial condition*.

General Solution ContD

- $m = 0 \quad \longleftrightarrow \quad$ Algebraic/Polynomial equation;
  $y = P(x) \longleftrightarrow \quad y = \varphi(x)$.
- $m = 1 \quad \longleftrightarrow \quad$ 1st Order ODE.;
  $y' = f(x, y) \quad \longleftrightarrow \quad y = \varphi(x, c)$.
- $m = 2 \quad \longleftrightarrow \quad$ 2nd Order ODE;
  $y'' = f(x, y, y') \quad \longleftrightarrow \quad y = \varphi(x, c_1, c_2)$.
- $m = 3 \quad \longleftrightarrow \quad$ 3rd Order ODE;
  $y''' = f(x, y, y', y'') \quad \longleftrightarrow \quad y = \varphi(x, c_1, c_2, c_3). \vdots$
- $m = k \quad \longleftrightarrow \quad k^{th}$ Order ODE;
  $y^{(k)} = f(x, y', \ldots, y^{(k-1)}) \quad \longleftrightarrow \quad y = \varphi(x, c_1, c_2, \ldots, c_k)$.

Example One

a.

$$y' = \cos x. \implies \int dy = \int \cos x dx$$
$$\implies y = \sin x + c.$$

$$y'' = x \text{ i.e. } \frac{dy'}{dx} = x \implies \int dy' = \int x dx$$
$$\implies y' = \frac{x^2}{2} + c_1, \implies y = \frac{x^3}{6} + c_1 x + c_2.$$

The constants, c_1, c_2 can be found, assuming the initial conditions are given as, say $x_0 = 1$, $y_0 = 1$ and $y_0' = 2$.

b. $y'' + y = 0$ has the solution $y = \sin x$.
Then,

$$(sin)'' + \sin x = (\cos x)' + \sin x = -\sin x + \sin x = 0.$$

Note:

$$y = \frac{1}{3}\sin x,\ y = \cos x,\ y = 2\cos x$$

are also solutions integrals, while

$$y = \sin x + \frac{1}{2} \text{ is not.}$$

c(i.). $yy' = -x$ has the implicit solution $y^2 + x^2 - 1 = 0$, $(y > 0)$ on $-1 < x < 1$.

c(ii.) $y'^2 = -1$ has no solution for $y \in \mathcal{R}$.

c(iii.) $|y'| + |y| = 0$ has no general solution, except the trivial solution y=0, $\forall x \in \mathcal{R}$.

c(iv.) $y'^2 - xy' + y = 0$ has general solution $y = cx - c^2$; and the singular solution $y = \frac{1}{4}x^2$.

MTH 201- MATHEMATICAL METHODS I

DIFFERENTIAL EQUATIONS LECTURE 003 First Order ODE

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1st Order ODE

- The standard form of a 1st ordinary differential equation is often given as

$$y' = f(x, y). \tag{1.3}$$

- This can manifest in various forms. Indeed,

$$
\begin{aligned}
y' &= f(x, y): \\
y' &= 2x^3 + 4; \\
y' &= 3x + y; \\
xy' &= y^2; \\
y'^2 &= x^3; \\
y' &= \sin x.
\end{aligned} \tag{1.4}
$$

•

$$y' = g(x)h(y):$$
$$y' = \frac{x}{y};$$
$$y' = xe^{x+y^2};$$
$$(x^2+1)y' + xy = 0. \tag{1.5}$$

$$A(x,y)dx + B(x,y)dy = 0; \frac{\partial A}{\partial y} = \frac{\partial B}{\partial x}:$$
$$(2x+y-3)dx + (x-4y+1)dy;$$
$$(2x+3x^2y)dx + (x^3-3y^2)dy. \tag{1.6}$$

We say the 1st oder ode (1.3) or (1.4) is in *explicit* form, (1.5) is in *separable variable* form while (1.6) is in *exact/total* form. Meanwhile, each of these is deducible from the *implicit* form $\Phi(x, y, y'(x)) = 0$.

Linear 1st Order ODE

- A linear 1st order ode is given by

$$\frac{dy}{dx} + P(x)y = Q(x). \tag{2.1}$$

- This is equivalent to replacing the right hand side of (1.3) with the expression $Q(x) - P(x)y$, i.e. making the substitution

$$f(x, y) \equiv Q(x) - P(x)y.$$

- The equation (2.1) falls into the category of the so-called *inexact* ode, which is solved by the *Integrating Factor method* (IFM). This so-called *factor*, $\mu(x)$ is a function of the independent variable x only. And the *philosophy* in the *method* is that when the factor multiplies an *inexact equation*, it converts it to an *exact* one.

- Thus, multipy (2.1) by $\mu(x)$

$$\mu(x)\frac{dy}{dx} + \mu(x)P(x)y = \mu(x)Q(x) \qquad (2.2)$$

and impose the condition on $\mu(x)$ that it must convert the left hand side of (2.2) to *an exact* (i.e. *a total differential*) expression:

$$\mu(x)\frac{dy}{dx} + \mu(x)P(x)y = \frac{d}{dx}[\mu(x)y(x)] \qquad (2.*)$$

- But, by the rule of differentiating product of functions, the rhs of (2.*) is

$$\overset{(*)}{\Longrightarrow} \frac{d}{dx}[\mu(x)y(x)]$$

$$= \mu(x)\frac{dy}{dx} + y(x)\frac{d\mu(x)}{dx}. \qquad (2.3$$

$$\overset{(2),(*),(3)}{\Longrightarrow} \quad \mu(x)\frac{dy}{dx} + y(x)\frac{d\mu(x)}{dx}$$

$$= \frac{d}{dx}[\mu(x)y(x)] = \mu(x)\frac{dy}{dx} + \mu(x)P(x)y(x). \tag{2.4}$$

Comparing terms in (2.4) gives

$$\overset{(2.4)}{\Longrightarrow} \frac{d\mu(x)}{dx} = \mu(x)P(x);$$

$$\Longrightarrow \quad \frac{d\mu(x)}{\mu} = P(x)dx \quad \Longrightarrow \quad \mu(x) = e^{\int P(x)dx}. \tag{2.5}$$

- Thus the integrating factor is deduced as

$$\boxed{\mu(x) = e^{\int P(x)dx}}. \tag{2.6}$$

- Now, another implication of condition (*) is that from (2.2)

$$\frac{d}{dx}[\mu(x)y(x)] = \mu(x)Q(x). \implies \mu(x)y(x) = \int \mu(x)Q(x)dx;$$

$$\boxed{y(x) = \tfrac{1}{\mu(x)} \int \mu(x)Q(x)dx}. \tag{2.7}$$

- **NB:** It is noted that the solution of the linear equation (1) is not limited to the IFM. Indeed, it can be solved using the substitution

$$y(x) = u(x)v(x), \tag{2.8}$$

which will transform equation (1) to the form

$$\left[\frac{du}{dx} + P(x)u\right] v + \frac{dv}{dx} u = Q(x). \tag{2.1$'$}$$

Then, in a manner similar to (2.*), impose the condition

$$\frac{du}{dx} + P(x)u = 0, \tag{2.**}$$

which, as an homogeneous equation now, is readily solved for u; and by implication from (2.1′) find v; and consequently by (2.8), obtain y.

Example 2a

- Solve for $y(x)$ if $\frac{dy}{dx} + 2xy = 4x$.
- **Solution:** This is a linear ode, degree one. It is an *inexact equation* and can be multiplied my an integrating factor, $\mu(x)$, so as to be converted to an *exact-type*. Here, $P(x) = 2x$ and $Q(x) = 4x$.

$$\overset{(2.6)}{\Longrightarrow} \quad \mu(x) = e^{\int P(x)dx} = e^{\int 2xdx}.$$

$$\overset{(2.7)}{\Longrightarrow} \quad \mu(x)y(x) = \int \mu(x)Q(x)dx \implies e^{x^2}y(x) = \int e^{x^2}4xdx.$$

$$\implies e^{x^2}y(x) = 2\int e^{x^2}dx^2 \implies e^{x^2}y(x) = 2e^{x^2}x^2 + C.$$

$$\implies y(x) = 2 + Ce^{-x^2}.$$

Example 2b

- Find $y(x)$ if $y' + y\cot x = 1$.
- **Solution:** Again, this is a linear ode, degree one. We solve it by the IFM. $P(x) = \cot x$ and $Q(x) = 1$.

$$\overset{(2.6)}{\Longrightarrow} \quad \mu(x) = e^{\int P(x)dx} = e^{\int \cot x dx}.$$

$$\Longrightarrow \quad \mu(x) = e^{\int \left(\frac{d\cos x}{\sin x}\right)} = e^{ln|\sin x|} = sinx.$$

$$\overset{(2.7)}{\Longrightarrow} \quad \mu(x)y(x) = \int \mu(x)Q(x)dx \;\Longrightarrow\; \sin x\, y(x) = \int \sin x dx.$$

$$\Longrightarrow \sin x\, y(x) = -\int d\cos x \;\Longrightarrow\; \sin x\, y(x) = -\cos x + C,$$

$$\Longrightarrow \quad y(x) = -\cot x + C\, cosec\, x.$$

MTH 201- MATHEMATICAL METHODS I

DIFFERENTIAL EQUATIONS LECTURE 004

Nonlinear ODE Reducible to Linear Ones

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Bernoulli Equation

- The nonlinear equation

$$\frac{dy}{dx} + P(x)y = Q(x)y^n; \quad n \neq 0, 1 \tag{3.1}$$

is called the *Bernoulli equation*, named after the Swiss Mathematician, Jacobi Bernoulli.

- When $n = 0$, the equation reduces to the linear one (2.1); and when $n = 1$ it also reduces to a linear one, but without a right hand side (i.e. in (2.1) $Q \equiv 0$).

- However, the nonlinear equation ($n > 1$) can be transformed to a linear one, by dividing equation (1) with y^n and then making the substitution $v = y^{1-n}$

$$y^{-n}\frac{dy}{dx} + P(x)y^{1-n} = Q(x).$$

$$\frac{dv}{dx} + (1-n)P(x)v(x) = (1-n)Q(x); \quad v \equiv y^{1-n}, \;\; y(x) \neq 0. \tag{3.2}$$

- It is clear, (3.2) is now a linear equation with respect to $v(x)$ and can be solved by the IFM described in respect of (3.1).

Example 3a

- Find $y(x)$, if $\dfrac{dy}{dx} + \dfrac{y}{x} = x^3y^4$.
- **Solution:**
  Here, by (1), $n = 4 \implies v = y^{1-4} = y^{-3}$.

$$\implies \frac{dy}{dx} = -\frac{y^4}{3}\frac{dv}{dx}.$$

Put this in the given equation and rearrange, to obtain

$$\frac{dv}{dx} - \frac{3v}{x} = -6x^3.$$

This is now a linear equation, which can be solved by the IFM:

$$P = -\frac{3}{x}, \ \ Q = -6x^; \ \mu = e^{\int Pdx}, \ \mu v = \int \mu Q dx$$

- $$\implies e^{-3\int \frac{dx}{x}} = e^{-3\ln x} = e^{\ln x^{-3}}. \implies \mu = \frac{1}{x^3}.$$

$$\mu v = \int \mu Q dx \implies \frac{1}{x^3} v = \int \frac{1}{x^3}(-6x^3) dx;$$

$$\frac{1}{x^3} v = -\int 6 dx = -6x + C;$$

$$\implies v = -6x^4 + Cx^3.$$

$$i.e. \quad y^{-3} = -6x^4 + Cx^3.$$

NB: Since we had to divide by y^n, then $y = 0$ is a singular solution of the Bernoulli equation, always.

Example 3b

- Solve for y, if $\dfrac{dy}{dx} + \dfrac{y}{x} = x^{-2}y^2$.
- **Solution:**
  Here, by (1) $n = 2 \implies v = y^{1-2} = y^{-1}; \quad \frac{dv}{dx} = -y^{-2}\frac{dy}{dx}$.
  Reflect this in the given equation and rearrange, to obtain

$$\implies -\frac{dv}{dx} + \frac{v}{x} = \frac{1}{x^2}, \; i.e. \; \frac{dv}{dx} - \frac{v}{x} = -x^{-2}.$$

$$\mu = e^{\int P dx} = e^{\int \frac{1}{x} dx} = e^{-\ln x} = \frac{1}{x},$$

Riccati Equation

- The nonlinear equation

$$\frac{dy}{dx} + P(x)y = Q(x)y^2 + H(x), \tag{3.3}$$

which is quadratic in the unknown $y(x)$, is known as the *Riccati equation*, named after the Italian Venetian Mathematician, Jacopo Francesco Riccati (1676-1754). It is a special case of the Bernoulli equation (3.1), when $n = 2$.

- The *Riccati equation*

$$\frac{dy}{dx} = x^3(y-x)^2 + \frac{y}{x} \tag{3.4}$$

(i) has a solution $y = x$, (ii) becomes a *Bernoulli equation* by a substitution $w = y - x$, and consequently be so-solved.
- The Riccati equation (3.3)
(i) becomes a Bernoulli equation when $H \equiv 0$,
(ii) is reducible to the Bernoulli equation if be known a solution $y = v$, by setting $w = y - v$.

Riccati Equation ContD

- Taking x as a known solution of (3.4) (i.e. $y = x$), and put

$$y = w + x$$

in (3.4), noting that $y' = w' + 1$, gives

$$\stackrel{(3.4)}{\Longrightarrow} \quad w' + 1 = x^3(w)^2 + 1 + \frac{w}{x};$$

$$\Longrightarrow \quad w' - \frac{1}{x}w = x^3(w)^2.$$

$$\stackrel{\text{By Bernoulli (3.1)}}{\Longrightarrow} \quad P \equiv -\frac{1}{x}, \; Q \equiv x^3;$$

$$n = 2 \longleftrightarrow \text{Ricatti eqn (3.3)}, \; H \equiv 0.$$

- $$\overset{\text{By IFM}}{\Longrightarrow} \quad \mu = e^{\int P dx}; \quad \mu w = \int \mu Q dx.$$

$$\Longrightarrow \mu = e^{-\int \frac{1}{x} dx} = e^{-\ln x} = e^{\ln \frac{1}{x}} = \frac{1}{x}.$$

$$\frac{1}{x} w = \int \frac{1}{x} x^3 dx; \quad \frac{1}{x} w = \int x^2 dx; \quad \frac{1}{x} w = \frac{1}{3} x^3 + C.$$

$$w = \frac{1}{3} x^4 + Cx. \Longrightarrow y = w + x = \frac{1}{3} x^4 + (C + 1)x.$$

- Now, when the known solution of the Riccati equation (3.3) is the function $v(x)$, then putting into (3.3)

$$y = w + v$$

reduces the equation to

$$w' + v' + P(w + v) = Q(w + v)^2 + H.$$

$$\Longrightarrow \quad w' + v' + Pw + Pv = Qw^2 + Qv^2 + 2Qwv + H,$$

$$w' + Pw - 2Qvw + v' + Pv = Qw^2 + Qv^2 + H,$$

$$w' + (P - 2Qv)w - Qw^2 + v' + Pv = Qv^2 + H,$$

$$w' + (P + Q*)w - Qw^2 + v' + Pv = Qv^2 + H;$$

$$Q* \equiv -2Qv.$$

- Since, v is a known solution of Riccati equation

$$\implies v' + Pv = Qv^2 + H, \text{ i.e. } -v' - Pv + Qv^2 + H = 0.$$

$$\implies w' + P * w = Qw^2, \; P* \equiv P + Q*,$$

a *Bernoulli equation* for $n = 2$, which is solvable for w by the IFM, already well enunciated above. Consequently the solution $y = v + w$ is established.

MTH 201- MATHEMATICAL METHODS I

DIFFERENTIAL EQUATIONS LECTURE 005

Higher Order ODE Reducible to Lower Order Ones

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Higher Order ODEs

- Some ordinary differential equations of high order can be reduced to equations of lower order.
  Recall the *m-order* ode

$$\Phi(x, y', y'', ..., y^{(m)}) = 0. \qquad (4.1)$$

  Cases when the *order* can be reduced (i.e. *lowered*) include the following:

- **When y is absent in equation (4.1)**

$$\Phi(x, y', y'', ..., y^{(m)}) = 0. \qquad (4.2)$$

  Then, to reduce the order of equation in such a case make the *substitution*

$$p = y^{(k)}, \text{ where } k < m. \qquad (4.3)$$

  That is, we take as the *new unknown*, a lower order derivative of the dependent variable in the equation.

Example

- Determine the solution path, when $m = 2$ and y is absent such that we have from (2)

$$\Phi(x, y', y'') = 0. \tag{4.2'}$$

- **Solution**
  Since $m = 2$, a lower order is y'. So, as the new variable, let

$$p = y', \text{ i.e. } p = \frac{dy}{dx}. \tag{4.3'}$$

  This leads to the pair of equations

$$y' = p(x) \text{ and } y'' = p'. \text{ i.e. } \Phi(x, p, p') = 0. \tag{4.2''}$$

- We thus obtained a 1st order equation, which is solved for p; and then integrate p to obtain the original unknown, y.

Example

- Solve for y in the equation

$$y'y'' = 1. \tag{4.2'''}$$

- **Solution** Let

$$y' = p.$$

$$\stackrel{4.2'')}{\Longrightarrow} \quad pp' = 1.$$

The order of equation has thus become lowered/reduced; from *order-2* to *order-one*. i.e.

$$yy'' = 1 \quad \stackrel{4.2''')}{\Longrightarrow} \quad pp' = 1 \text{ or } p' = \frac{1}{p}.$$

- $$\implies pdp = dx \implies \frac{1}{2}dp^2 = dx. \implies dp^2 = 2dx$$
$$\implies p^2 = 2x + C,$$
$$\implies p = \pm\sqrt{2x + C}.$$
$$\text{But, } y' = p, \implies y = \int pdx. \implies y = \int (2x + C)^{\frac{1}{2}} dx$$
$$= \frac{1}{2}\int (2x + C)^{\frac{1}{2}} d(2x + C) = \frac{1}{2}\int (z)^{\frac{1}{2}} dz = \frac{1}{3}z^{\frac{3}{2}} + K;$$
$$z \equiv 2x + C.$$

Note that C and K are constants of integration.

Control Problem

- Solve the equation and compare with the indicated solution in square bracket
- $$y^3 y'' = 1. \quad \longleftrightarrow \quad C_1 y^2 - 1 = (C_1 x + C_2)^2,$$
  $$C_1, C_2 \quad \text{constants of integration.}$$
- $$y^2 y'' = y'^2. \quad \longleftrightarrow \quad C_1 - C_1^2 = \ln|C_1 x + 1| + C_2,$$
  $$C_1, C_2 \quad \text{constants of integration.}$$

When x is absent in equation (4.1)

- Equation (4.1) presents in the form

$$\Phi(y, y', y'', ..., y^{(m)}) = 0. \tag{4.4}$$

Then, we can reduce the order of equation in such a case with the *substitution*

$$y' = p(y), \text{ where } k < m. \tag{4.5}$$

i.e. making y the new independent variable.

Example

- Solve the equation

$$2yy'' = y'^2 + 1. \tag{4.4'}$$

- **Solution**

Here, $m = 2$, x is absent $\implies \Phi(y, y', y'') = 0$. So, take

$$y' = p(y), \; y'' = \frac{dp(y)}{dx} = \frac{dp(y)}{dy}\frac{dy}{dx} = pp'. \tag{4.5'}$$

Now, put $y' = p, \quad y'' = pp'$ in the original equation (4′)

$$2ypp' = p^2 + 1. \tag{4.5''}$$

Thus, the *order* of equation is lowered/reduced from <u>two</u> to <u>one</u>. Consequently, we solve equation (4.5″) to obtain

$$p = \pm\sqrt{C_1 y - 1}, \quad \text{i.e} \quad y' = \pm\sqrt{C_1 y - 1}.$$

$$\implies 4(C_1 y - 1) = C_1^2(x + C_2).$$

Example

- Solve the equation

$$y'' = yy'. \qquad (4.4'')$$

Here, $m = 2$, x is absent $\implies \Phi(y, y', y'') = 0$. So, follow $(4.5), (4.5')$ we have

$$\frac{dp(y)}{dy}. \implies p(y) = \frac{1}{2}y^2 + C_1. \text{ i.e } \frac{dy}{dx} = \frac{1}{2}y^2 + C_1.$$

$$\implies \frac{dy}{y^2 + 2C_1} = \frac{1}{2}dx.$$

$$\implies x = 2\int \frac{dy}{y^2 + \beta^2}, \ \beta \equiv \sqrt{2C_1} \implies x = 2\frac{1}{\beta}\tan^{-1}\frac{y}{\beta} + C_2.$$

NB: C_1, C_2 are constants of integration, while the table integral, $\int \frac{du}{u^2+k^2} = \frac{1}{k}\tan^{-1}\frac{u}{k}$, has been invoked.

Other Cases

- **When** y **&** y', y'', ..., $y^{(m)}$, **in equation (1), are homogeneous functions** i.e. if we replace y & y', y'', ..., $y^{(m)}$ with ky & ky', ky'', ..., $ky^{(m)}$, equation (1) will not change. The order of the equation is also reducible in this case by the substitution

$$y' = yv,$$

where v is a new unknown function.
- **When an equation is transformable, such that every term in the equation becomes an exact differential**
- **When** x **and** y **are homogeneous in the generalised sense:** i.e. if x and y are replaced with kx and $k^m y$, such that the derivatives change to the form

$$k^{m-1}y',\ k^{m-2}y'',\ k^{m-3}y''',\ \dots,\ k^{m-q}y^{q},\ \dots$$

MTH 201
Differential Equation III

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Introduction

Recall that differential equations originate from the mathematical formulation of a great variety of problems in science and engineering. In this module we consider problems that give rise to some of the types of first order ordinary differential equations studied in the previous lectures. First, we formulate the problem mathematically, thereby obtaining a differential equation. Then we solve the equation and attempt to interpret the solution in terms of the quantities involved in the original problem.
Before we apply our knowledge of differential equations to certain problems in mechanics, let us briefly recall certain principles of that subject. The **momentum** of a body is defined to be the product mv of its mass m and its velocity v. The velocity v and momentum are vector quantities. We now state the following basic law of mechanics:

Newton's Second Law.

The time rate of change of momentum of a body is proportional to the resultant force acting on the body and is in the direction of this resultant force.

In mathematical language, this law states that

$$\frac{d}{dt}(mv) = KF$$

where m is the mass of the body, v is its velocity, F is the resultant force acting upon it, and K is a constant of proportionality. If the mass m is considered constant, this reduces to

$$m\frac{dv}{dt} = KF, \quad \text{or} \quad a = K\frac{F}{m}, \quad \text{or} \quad F = kma$$

where $k = 1/K$ and $a = dv/dt$ is the acceleration of the body. The magnitude of the constant of proportionality k depends upon the units employed for force, mass, and acceleration. Obviously the simplest systems of units are those for which $k = 1$. When such a system is used then the last equation reduces to

$$F = ma. \tag{1}$$

Several systems of units for which $k = 1$ are in use. In this study we shall use only three:

1. The British gravitational system (British);
2. The centimeter-gram-second system (*cgs*); and
3. The meter-kilogram-second system (*mks*).

We summarize the various units of these three systems in Table 1

Quantities	British System	*cgs* System	*mks* System
force	pound	dyne	newton
mass	slug	gram	kilogram
distance	foot	centimeter	meter
time	second	second	second
acceleration	ft/sec^2	cm/sec^2	m/sec^2

Table 1

Let us now apply Newton's second law to a freely falling body (a body falling toward the earth in the absence of air resistance). Let the mass of the body be m and let w denote its weight. The only force acting on the body is its weight and so this is the resultant force. The acceleration is that due to gravity, denoted by g, which is approximately $32 ft/sec^2$ in the *British system*, $980 cm/sec^2$ in the *cgs system*, and $9.8 m/sec^2$ in the *mks system* (for points near the earth's surface).

Newton's second law $F = ma$ thus reduces to $w = mg$. Thus

$$m = \frac{w}{g}. \tag{2}$$

Let us now consider a body B in rectilinear motion, that is, in motion along a straight line L. On L we choose a fixed reference point as origin O, a fixed direction as positive, and a unit of distance. Then the coordinate x of the position of B from the origin O tells us the distance or displacement of B. (See Figure 1)

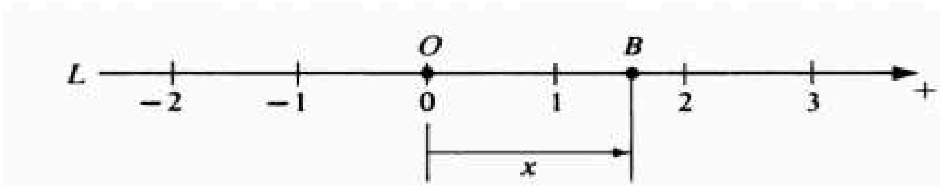

Figure: 1. Rectilinear Motion.

We have the following about the motion of B along L :
(i) The **instantaneous velocity** of B is the time rate of change of x :

$$v = \frac{dx}{dt}.$$

(ii) The **instantaneous acceleration** of B is the time rate of change of velocity v :

$$a = \frac{dv}{dt} = \frac{d^2x}{dt^2}.$$

Note that x, v, and a are vector quantities. All forces, displacements, velocities, and accelerations in the positive direction on L are positive quantities; while those in the negative direction are negative quantities. If we now apply Newton's second law $F = ma$ to the motion of B along L, noting that

$$\frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx},$$

we may express the law in any of the following forms:

$$m\frac{dv}{dt} = F, \tag{3}$$

$$m\frac{d^2x}{dt^2} = F, \tag{4}$$

$$mv\frac{dv}{dx} = F, \tag{5}$$

where F is the resultant force acting on the body. The form to use depends upon the way in which F is expressed.

Falling Body Problems

We shall now consider some examples of a body falling through air toward the earth. In such a circumstance the body encounters air resistance as it falls. The amount of air resistance depends upon the velocity of the body, but no general law exactly expressing this dependence is known. In some instances the law $R = kv$ appears to be quite satisfactory, while in others $R = kv^2$ appears to be more exact. In any case, the constant of proportionality k in turn depends on several circumstances. In the examples that follow we shall assume certain reasonable resistance laws in each case.

Example

A body weighing 8 *lb* falls from rest toward the earth from a great height. As it falls, air resistance acts upon it, and we shall assume that this resistance *(in pounds)* is numerically equal to $2v$, where v is the velocity *(in feet per second)*. Find the velocity and distance fallen at time t seconds.

Formulation.

We choose the positive x axis vertically downward along the path of the body B and the origin at the point from which the body fell. The forces acting on the body are:

1. F_1 its weight, 8 *lb*, which acts downward and hence is positive; and
2. F_2, the air resistance, numerically equal to $2v$, which acts upward and hence is the negative quantity $-2v$.

See Figure 2, where these forces are indicated. Newton's second law, $F = ma$, becomes

$$m\frac{dv}{dt} = F_1 + F_2$$

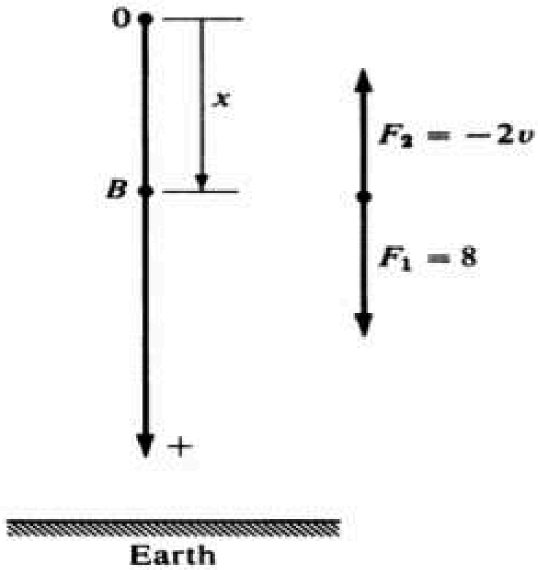

Figure: 2. Falling Body B.

Taking $g = 32$ and using $m = w/g = \frac{1}{4}$,

$$\frac{1}{4}\frac{dv}{dt} = 8 - 2v. \tag{6}$$

Since the body was initially at rest, we have the initial condition

$$v(0) = 0. \tag{7}$$

Separating the variables in (6) we have

$$\frac{dv}{8 - 2v} = 4\,dt.$$

Integrating both sides gives

$$8 - 2v = c_1 e^{-8t}.$$

Applying the condition (7) we find $c_1 = 8$. Thus the velocity at time t is given by

$$v = 4\left(1 - e^{-8t}\right). \tag{8}$$

Now to determine the distance fallen at time t, we write (8) in the form

$$\frac{dx}{dt} = 4\left(1 - e^{-8t}\right)$$

and note that $x(0) = 0$. Integrating the above equation, we obtain

$$x = 4\left(t + \frac{1}{8}e^{-8t}\right) + c_2.$$

Since $x = 0$ when $t = 0$, we find $c_2 = -\frac{1}{2}$ and hence the distance fallen is given by

$$x = 4\left(t + \frac{1}{8}e^{-8t} - \frac{1}{8}\right). \tag{9}$$

Interpretation of Results. Equation (8) shows us that as $t \to \infty$, the velocity v approaches the limiting velocity $4(ft/sec)$. We also observe that this limiting velocity is approximately attained in a very short time. Equation (9) states that as $t \to \infty$, $x \to \infty$ also. Does this imply that the body will plow through the earth and continue forever? Of course not; for when the body reaches the earth's surface its motion will certainly cease and the differential equation (6) and hence equation (9) no longer apply.

Frictional Forces

If a body moves on a rough surface, it will encounter not only air resistance but also another resistance force due to the roughness of the surface. This additional force is called **friction.** It is shown in physics that the friction is given by μN, where

1. μ is a constant of proportionality called the **coefficient of friction,** which depends upon the roughness of the given surface; and
2. N is the normal (that is, perpendicular) force which the surface exerts on the body.

We now apply Newton's second law to a problem in which friction is involved.

Example

An object weighing 48 *lb* is released from rest at the top of a plane metal slide that is inclined 30° to the horizontal. Air resistance (in pounds) is numerically equal to one-half the velocity (in feet per second), and the coefficient of friction is one-quarter.

1. What is the velocity of the object 2 *sec* after it is released?
2. If the slide is 24 *ft* long, what is the velocity when the object reaches the bottom?

Formulation. The line of motion is along the slide. We choose the origin at the top and the positive x direction down the slide. If we temporarily neglect the friction and air resistance, the forces acting upon the object A are:

1. Its weight, 48 *lb*, which acts vertically downward; and
2. The normal force, N, exerted by the slide which acts in an upward direction perpendicular to the slide. (See Figure 3)

The components of the weight parallel and perpendicular to the slide have magnitude $48 \sin 30° = 24$ and $48 \cos 30° = 24\sqrt{3}$, respectively.

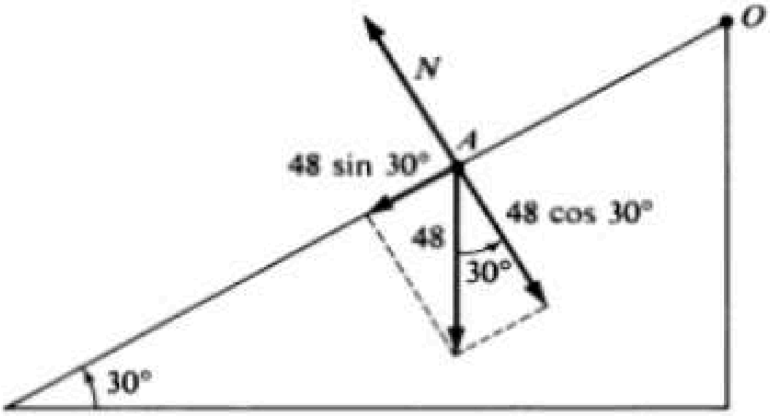

Figure: 3. Motion of object A.

The components perpendicular to the slide are in equilibrium and hence the normal force N has magnitude $24\sqrt{3}$. Now, taking into consideration the friction and air resistance, we see that the forces acting on the object as it moves along the slide are the following:

1. F_1, the component of the weight parallel to the plane, having numerical value 24. Since this force acts in the positive (downward) direction along the slide, we have
$$F_1 = 24;$$
2. F_2, the frictional force, having numerical value $\mu N = \frac{1}{4}(24\sqrt{3})$. Since this acts in the negative (upward) direction along the slide, we have
$$F_2 = -6\sqrt{3};$$
3. F_3, the air resistance, having numerical value $\frac{1}{2}v$. Since $v > 0$ and this also acts in the negative direction, we have
$$F_3 = -\frac{1}{2}v.$$

Apply Newton's second law $F = ma$, where

$$F = F_1 + F_2 + F_3 = 24 - 6\sqrt{3} - \frac{1}{2}v \text{ and } \frac{w}{g} = \frac{48}{32} = \frac{3}{2}.$$

Thus the differential equation is

$$\frac{3}{2}\frac{dv}{dt} = 24 - 6\sqrt{3} - \frac{1}{2}v. \tag{10}$$

Since the object is released from rest, the initial condition is

$$v(0) = 0. \tag{11}$$

Separating the variables we find

$$\frac{dv}{48 - 12\sqrt{3} - v} = \frac{dt}{3}.$$

Integrating and simplifying, we have

$$v = 48 - 12\sqrt{3} - c_1 e^{-t/3}.$$

Condition (11) gives $c_1 = 48 - 12\sqrt{3}$. Hence we have

$$v = (48 - 12\sqrt{3})(1 - e^{-t/3}) \tag{12}$$

Question 1 is answered by letting $t = 2$ in equation (12). we find

$$v(2) = (48 - 12\sqrt{3})(1 - e^{-2/3}) \approx 13.2\ ft/sec.$$

To answer Question 2, integrating equation (12) to get

$$x = (48 - 12\sqrt{3})(t + 3e^{-t/3}) + c_2$$

Since $x(0) = 0$ it follows that $c_2 = -3(48 - 12\sqrt{3})$. Thus the distance cover at time t is

$$x = (48 - 12\sqrt{3})(t + 3e^{-t/3} - 3).$$

Since the slide is 24 ft long, the object reaches the bottom at the time T determined from the transcendental equation

$$24 = (48 - 12\sqrt{3})(T + 3e^{-T/3} - 3) \text{ or } 3e^{-T/3} = \frac{47 + 2\sqrt{3}}{13} - T.$$

The value of T that satisfies this equation is approximately 2.6. Thus from equation (12) the velocity of the object when it reaches the bottom is given approximately by

$$(48 - 12\sqrt{3})(1 - e^{-0.9}) \approx 16.2 \ ft/sec.$$

Rate Problems

In certain problems the rate at which a quantity changes is a known function of the amount present and/or the time, and it desired to find the quantity itself. If x denotes the amount of the quantity present at time t, then dx/dt denotes the rate at which the quantity changes and we are at once led to a differential equation.

Example

The rate at which radioactive nuclei decay is proportional to the number of such nuclei that are present in a given sample. Half of the original number of radioactive nuclei have undergone disintegration in a period of 1500 years.

1. What percentage of the original radioactive nuclei will remain after 4500 years?
2. In how many years will only one-tenth of the original number remain?

Formulation. Let x be the amount of radioactive nuclei present after t years. Then dx/dt represents the rate at which the nuclei decay. Since the nuclei decay at a rate proportional to the amount present, we have

$$\frac{dx}{dt} = Kx, \tag{13}$$

where K is a constant of proportionality. The amount x is clearly positive; further, since x is decreasing, $dx/dt < O$. Thus, from Equation (13), we must have $K < O$. In order to emphasize that x is decreasing, we prefer to replace K by a positive constant preceded by a minus sign. Thus we let $k = -K > 0$ and write the differential equation (13) in the form

$$\frac{dx}{dt} = -kx. \tag{14}$$

Suppose x_0 denote the amount initially present, we also have the initial condition

$$x(0) = x_0. \tag{15}$$

We know that we shall need such a condition in order to determine the arbitrary constant that will appear in a one-parameter family of solutions of the differential equation (14). From the statement of the problem, for we are told that half of the original number disintegrate in 1500 years. Thus half also remain at that time, and this at once gives the condition

$$1500x = \frac{1}{2}x_0. \tag{16}$$

Separating variables in (14), integrating, and simplifying, we have at once

$$x = ce^{-kt}.$$

Applying the initial condition (15), $x(0) = x_0$, we find that $c = x_0$ and hence we obtain

$$x = x_0 e^{-kt} \tag{17}$$

We have not yet determined k. Thus we now apply condition (16), $x = \frac{1}{2}x_0$ when $t = 1500$, to equation (17). We find

$$e^{-1500k} = \frac{1}{2} \text{ or } e^{-k} = (\frac{1}{2})^{1/1500}.$$

From this we find

$$k = \frac{\ln 2}{1500} \approx 0.00046.$$

Using this, (17) becomes

$$x = x_0 e^{-0.00046t}. \tag{18}$$

Example

Consider a population of field mice who inhabit a certain rural area. In the absence of predators we assume that the mouse population increases at a rate proportional to the current population.

If we denote time by t and the mouse population by $p(t)$, then the assumption about population growth can be expressed by the equation

$$\frac{dp}{dt} = rp, \tag{19}$$

where the proportionality factor r is called the **rate constant** or **growth rate.** To be specific, suppose that time is measured in months and that the rate constant r has the value $0.5/month$. Then each term in (19) has the units of mice/month. Now let us add to the problem by supposing that several owls live in the same neighbourhood and that they kill 15 field mice per day. To incorporate this information into the model, we must add another term to the differential equation (19), so that it becomes

$$\frac{dp}{dt} = 0.5p - 450. \tag{20}$$

Separating the variables we have

$$\frac{dp}{p-900} = \frac{1}{2}dt, \quad p \neq 900. \tag{21}$$

A parameter family of solutions is

$$|p-900| = e^c e^{t/2} \quad \text{or} \quad p-900 = \pm e^c e^{t/2} \quad \text{or} \quad p = 900 + ke^{t/2} \tag{22}$$

where $k = \pm e^c$ is also an arbitrary (non-zero) constant. Graphs of (22) for several values of k are shown in Figure 4

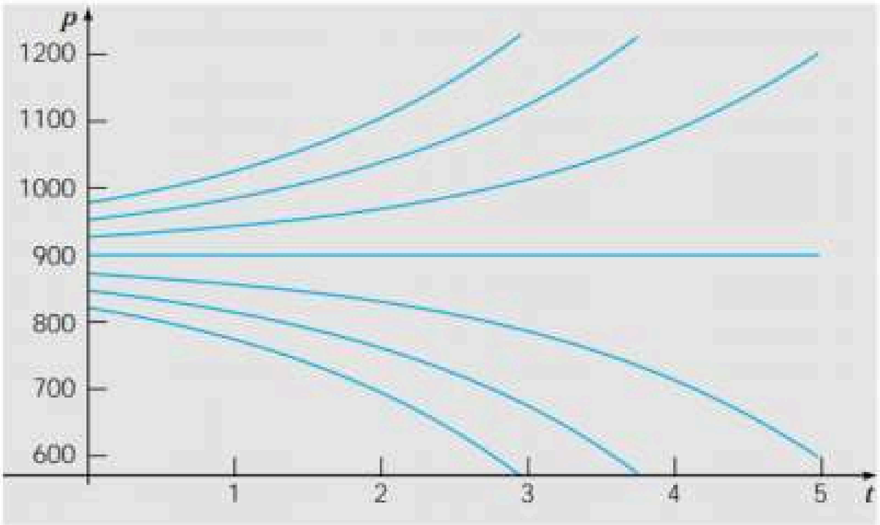

Figure: 4. Graphs of (22) for several values of k.

Exercises

1. A skydiver equipped with parachute and other essential equipment falls from rest toward the earth. The total weight of the man plus the equipment is 160 *lb*. Before the parachute opens, the air resistance (in pounds) is numerically equal to $\frac{1}{2}v$, where v is the velocity (in feet per second). The parachute opens 5 *sec* after the fall begins; after it opens, the air resistance (in pounds) is numerically equal to $\frac{5}{8}v$ where v is the velocity (in feet per second). Find the velocity of the skydiver
   1.1 before the parachute opens; and
   1.2 after the parachute opens.
2. A stone weighing 4 *lb* falls from rest toward the earth from a great height. As it falls it is acted upon by air resistance that is numerically equal to $\dfrac{1}{2}v$ (in pounds), where v is the velocity (in feet per second).
   2.1 Find the velocity and distance fallen at time t sec.
   2.2 Find the velocity and distance fallen at the end of 5 *sec*.
3. A ball weighing 3/4 *b* is thrown vertically upward from a point 6 *ft* above the surface of the earth with an initial velocity of 20 *ft*/*sec*. As it rises it is acted upon by air resistance that is numerically equal to $\frac{1}{64v}$ (in pounds), where v is the velocity (in feet per second). How high will the ball rise?

References

1. Haberman R. - Elementary Applied Partial Differential Equations with Fourier Series and Boundary Value Problems-(1987).
2. C. Henry Edwards, David E. Penney - Elementary Differential Equations-Prentice Hall (2007).
3. Henry Thomas Herbert Piaggio - An elementary treatise on differential equations and their applications- G. Bell and Sons, LTD. London (1929).
4. Morris Tenenbaum, Harry Pollard - Ordinary differential equations: an elementary textbook for students of mathematics, engineering, and the sciences-Dover Publications, (1985).
5. Patrick J. Murphy - Elementary Matrices and Some Applications to Dynamics and Differential Equations-Cambridge University Press (1960).
6. Shepley L. Ross - Introduction to Ordinary Differential Equations, 4th Edition-Wiley (1989).
7. William E. Boyce, Richard C. DiPrima - Elementary Differential Equations And Boundary Value Problems-Wiley (2000).

MTH 201
Differential Equation III

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Orthogonal Trajectories

The problem of finding the orthogonal trajectories of a given family of curves arises in many physical situations. For example, in a two-dimensional electric field the lines of force (flux lines) and the equipotential curves are orthogonal trajectories of each other.

Definition

Let

$$F(x, y, c) = 0 \tag{1}$$

be a given one-parameter family of curves in the xy Plane. A Curve that intersects the curves of the family (1) at right angles is called **an orthogonal trajectory** of the given family.

Example

Consider the family of circles

$$x^2 + y^2 = c^2 \tag{2}$$

with center at the origin and radius c. Each straight line through the origin,

$$y = kx, \tag{3}$$

is an orthogonal trajectory of the family of circles (2). Conversely, each circle of the family (2) is an orthogonal trajectory of the family of straight lines (3). The families (2) and (3) are orthogonal trajectories of each other. In Figure 1 several members of the family of circles (2), drawn solidly, and several members of the family of straight lines (3), drawn with dashes, are shown.

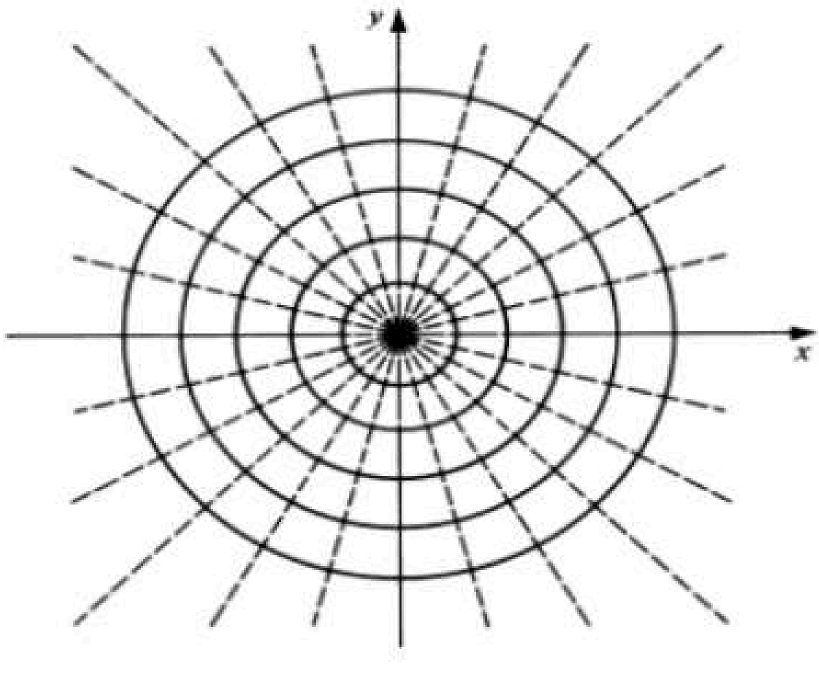

Figure: 1. Orthogonal trajectories.

Procedure for Finding the Orthogonal Trajectories of a Given Family of Curves

Step 1. From the equation

$$F(x, y, c) = 0$$

of the given family of curves, find the differential equation

$$\frac{dy}{dx} = f(x, y) \tag{4}$$

of this family.

Step 2. In the differential equation $dy/dx = f(x, y)$ so found in Step 1, replace $f(x, y)$ by its negative reciprocal $-1/f(x, y)$. This give the differential equation

$$\frac{dy}{dx} = -\frac{1}{f(x, y)} \tag{5}$$

of the orthogonal trajectories.

Step 3. Obtain a one-parameter family

$$G(x, y, c) = 0 \text{ or } y = F(x, c)$$

of solutions of the differential equation (5), thus obtaining the desired family of orthogonal trajectories (except possibly for certain trajectories that are vertical lines and must be determined separately).

Example

In Example 1 we stated that the set of orthogonal trajectories of the family of circles

$$x^2 + y^2 = c^2$$

is a family of straight line

$$y = kx.$$

Let us verify this using the above steps.

Step 1. Differentiating the equation

$$x^2 + y^2 = c^2$$

of the given family, we obtain

$$x + y\frac{dy}{dx} = 0.$$

From this we obtain the differential equation

$$\frac{dy}{dx} = -\frac{x}{y} \tag{6}$$

of the given family (2).

Step 2. We replace $-x/y$ by its negative reciprocal y/x in the differential equation (6) to obtain the differential equation

$$\frac{dy}{dx} = \frac{y}{x} \tag{7}$$

of the orthogonal trajectories.

Step 3. We now solve the differential equation (7). Separating variables, we have

$$\frac{dy}{y} = \frac{dx}{x},$$

integrating, we obtain (3) i.e.

$$y = kx.$$

This is a one-parameter family of solutions of the differential equation (7) and thus represents the family of orthogonal trajectories of the given family of circles (2).

Example

Find the orthogonal trajectories of the family of curves

$$cx^2 + y^2 = 1. \tag{8}$$

Differentiate (8) with respect to x we find

$$\frac{dy}{dx} = -\frac{cx}{y}. \tag{9}$$

From (8) we have

$$c = \frac{1 - y^2}{x^2}$$

so that

$$\frac{dy}{dx} = -\frac{(1 - y^2)}{xy} \tag{10}$$

Next, replace $-\frac{(1-y^2)}{xy}$ by its negative reciprocal i.e. $-\frac{xy}{y^2-1}$, the differential equation associated with this negative reciprocal is

$$\frac{dy}{dx} = -x\left(\frac{y}{y^2 - 1}\right). \tag{11}$$

Next, separating the variables and integrate to get

$$x^2 + y^2 - \ln y^2 = c$$

where $c = 2c_0$ is the one-parameter family of solutions of (11) and also the family of the orthogonal trajectories of equation (8).

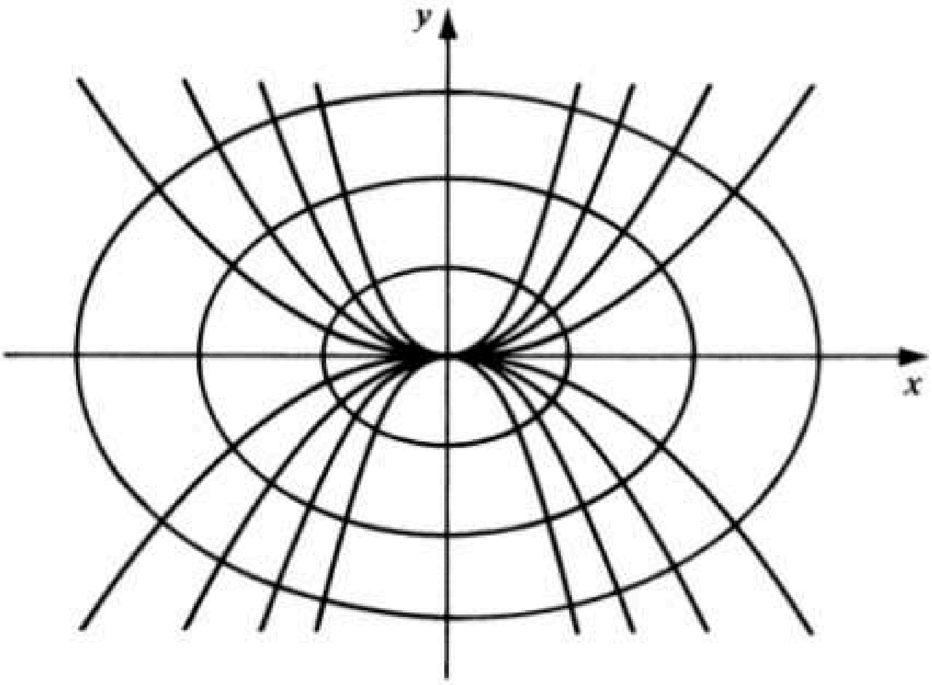

Figure: 2. Oblique trajectories.

Oblique Trajectories

Definition

Let

$$F(x, y, c) = 0 \tag{12}$$

be a one-parameter family of curves. A curve that intersects the curves of the family (12) at a constant angle $\alpha \neq 90°$ is called an **oblique trajectory** of the given family. See Figure 2

Suppose the differential equation of a family is

$$\frac{dy}{dx} = f(x, y). \tag{13}$$

Then the curve of the family (13) through the point (x, y) has slope $f(x, y)$ at (x, y) and hence its tangent line has angle of inclination $\tan^{-1}[f(x, y)]$ there. The tangent line of an oblique trajectory that intersects this curve at the angle α will thus have angle of inclination

$$\tan^{-1}[f(x, y)] + \alpha$$

at the point (x, y). Hence the slope of this oblique trajectory is given by

$$\tan\left\{\tan^{-1}[f(x, y)] + \alpha\right\} = \frac{f(x, y) + \tan\alpha}{1 - f(x, y)\tan\alpha}. \tag{14}$$

Example

Find a family of oblique trajectories that intersect the family of straight lines $y = cx$ at angle 45°.

Step 1. The derivative of $y = cx$ with respect to x is $dy/dy = c$. Eliminating c we find the differential equation

$$\frac{dy}{dx} = \frac{y}{x} = f(x, y) \tag{15}$$

of the given family of straight lines.

Step 2. We replace $f(x, y) = y/x$ in Equation (15) by

$$\frac{f(x,y) + \tan\alpha}{1 - f(x,y)\tan\alpha} = \frac{y/x + 1}{1 - y/x} = \frac{x + y}{x - y}.$$

($\tan\alpha = \tan 45^\circ = 1$). Thus the differential equation of the desired oblique trajectories is

$$\frac{dy}{dx} = \frac{x + y}{x - y}. \tag{16}$$

Step 3. We now solve the differential equation (16). Observing that it is a homogeneous differential equation, we let $y = vx$ to obtain

$$v + x\frac{dv}{dx} = \frac{x + y}{1 - v}$$

After simplifications this becomes

$$\frac{(v - 1)dv}{v^2 + 1} = -\frac{dx}{x}.$$

Integrating and replacing v by y/x, we obtain the family of oblique trajectories in the form

$$\ln c^2(x^2 + y^2) - 2\arctan\left(\frac{y}{x}\right) = 0.$$

Exercises

In Exercises $1-9$ find the orthogonal trajectories of each given family of curves. In each case sketch several members of the family and several of the orthogonal trajectories on the same set of axes.

1.	$y = cx^2$,	2.	$y^2 = cx$,
3.	$cx^2 + y^2 = 1$,	4.	$y = e^{cx}$,
5.	$y = x - 1 + ce^{-x}$,	6.	$y = \frac{cx^2}{x+1}$,
7.	$x^2 + y^2 = cx^2$,	8.	$x^2 = 2y - 1 + ce^{-2y}$
9.	$x = \frac{y^2}{4} + \frac{c}{y^2}$,	10.	$x^2 - y^2 = cx^2$.

11. A given family of curves is said to be self-orthogonal if its family of orthogonal trajectories is the same as the given family. Show that the family of parabolas $y^2 = 2cx + c^2$ is self orthogonal.
12. Find a family of oblique trajectories that intersect the family of circles $x^2 + y^2 = c^2$ at angle 45°.
13. Find a family of oblique trajectories that intersect the family of parabolas $y^2 = cx$ at angle 60°.
14. Find a family of oblique trajectories that intersect the family of curves $x + y = cx^2$ at angle θ such that $\tan\theta = 2$.

References

1. Robert Alexander Frazer, William Jolly Duncan, Arthur Roderich Collar - Elementary Matrices And Some Applications To Dynamics And Differential Equations - Cambridge University Press-(1947)
2. Haberman R. - Elementary Applied Partial Differential Equations with Fourier Series and Boundary Value Problems-(1987).
3. C. Henry Edwards, David E. Penney - Elementary Differential Equations-Prentice Hall (2007).
4. Henry Thomas Herbert Piaggio - An elementary treatise on differential equations and their applications- G. Bell and Sons, LTD. London (1929).
5. Morris Tenenbaum, Harry Pollard - Ordinary differential equations: an elementary textbook for students of mathematics, engineering, and the sciences-Dover Publications, (1985).
6. Patrick J. Murphy - Elementary Matrices and Some Applications to Dynamics and Differential Equations-Cambridge University Press (1960).
7. Shepley L. Ross - Introduction to Ordinary Differential Equations, 4th Edition-Wiley (1989).
8. William E. Boyce, Richard C. DiPrima - Elementary Differential Equations And Boundary Value Problems-Wiley (2000).

MTH 201
Differential Equation III

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SECOND ORDER ORDINARY DIFFERENTIAL EQUATIONS(ODES) WITH CONSTANT COEFFICIENTS

Introduction:

Linear equations of second order are of crucial importance in the study of differential equations for two main reasons:

1. The first is that linear equations have a rich theoretical structure that underlies a number of systematic methods of solution. Further, a substantial portion of this structure and these methods are understandable at a fairly elementary mathematical level; and
2. Secondly, second order linear equations are vital to any serious investigation of the classical areas of mathematical physics. One cannot go very far in the development of fluid mechanics, heat conduction, wave motion, or electromagnetic phenomena without finding it necessary to solve second order linear differential equations.

In the preceding modules we used the dy/dx notation to denote the derivative of a function $y = y(x)$ with respect to x. In what follows we shall generally use the prime $'$ notation to denote derivatives. Thus, for example, instead of writing

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - 4y = \sin x, \quad \text{we write} \quad y'' + 3y' - 4y = \sin x.$$

Homogeneous Equations with Constant Coefficients

A second order ODE has the form

$$y'' = f(x, y, y'), \tag{1}$$

where f is some given function. Usually, we will generally denote the independent variable by x, although time t is often the independent variable used in physical problems. Equation (1) is said to be **linear** if the function f has the form

$$f(x, y, y') = g(x) - q(x)y - p(x)y', \tag{2}$$

that is, if f is linear in y and y'. In (2) g, p, and q are specified functions of the independent variable x but do not depend on y. In this case we usually rewrite equation (1) in the form

$$y'' + p(x)y' + q(x)y = g(x). \tag{3}$$

We often write equation (1) in the form

$$P(x)y'' + Q(x)y' + R(x)y = G(x), \qquad P(x) \neq 0. \tag{4}$$

We can divide equation (4) by $P(x)$ and thereby obtain equation (3) with

$$p(x) = \frac{Q(x)}{P(x)}, \quad q(x) = \frac{R(x)}{P(x)}, \quad g(x) = \frac{G(x)}{P(x)}. \tag{5}$$

If equation (1) is not of the form (3) or (4), then it is called **nonlinear.** An initial value problem consists of a differential equation such as equation (1), (3), or (4) together with a pair of **initial conditions**

$$y(x_0) = y_0, \qquad y'(x_0) = y_0', \tag{6}$$

where y_0 and y_0' are given initial values. Note that the initial conditions for a second order equation prescribe a particular point (x_0, y_0) through which the graph of the solution must pass, and also the slope y_0' of the graph at that point. A second order linear equation is said to be **homogeneous** if the term $g(x)$ in equation (3), or the term $G(x)$ in equation (4), is zero for all x. Otherwise, the equation is called **nonhomogeneous.** We continue our discussion with homogeneous equations of the form

$$P(x)y'' + Q(x)y' + R(x)y = 0. \tag{7}$$

We consider equations in which the functions $P, Q,$ and R are constants.

In this case, equation (7) becomes

$$ay'' + by' + cy = 0 \tag{8}$$

where a, b, and c are given real constants. Now let us also seek exponential solutions of (8). Thus we suppose that

$$y = e^{rx},$$

where r is a parameter to be determined. Then it follows that

$$y' = re^{rx}$$

and

$$y'' = r^2 e^{rx}.$$

By substituting these expressions for y, y', and y'' in (8) we obtain

$$\left(ar^2 + br + c\right)e^{rx} = 0,$$

or since $e^{rx} \neq 0$ for all $x \in \mathbb{R}$, it follows

$$ar^2 + br + c = 0. \tag{9}$$

Equation (9) is called the **axillary or characteristic equation** for the homogeneous differential equation (8).

Case 1. Real and Distinct Roots.

Since (9) is a quadratic equation with real coefficients, it has two roots, which may be

1. real and distinct (different roots),
2. real but repeated (equal roots), or
3. complex conjugates (imaginary roots).

Case 1. Assuming that the roots of the characteristic equation (9) are real and different, let us denote the roots by r_1 and r_2, where $r_1 \neq r_2$. Then

$$y_1(x) = e^{r_1 x} \text{ and } y_2(x) = e^{r_2 x}$$

are two linearly independent solutions of (8). Since the linear combination of independent solutions is still a solution it now follows

$$y(x) = c_1 y_1(x) + c_2 y_2(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}. \tag{10}$$

Equation (10) is called **a general solution** or **a two parameters family of solution** of (8). To verify that this is so, differentiate equation (10), we find that

$$y' = c_1 r_1 e^{r_1 x} + c_2 r_2 e^{r_2 x} \tag{11}$$

and

$$y'' = c_1 r_1^2 e^{r_1 x} + c_2 r_2^2 e^{r_2 x}. \tag{12}$$

Substituting these expressions for y, y', and y'' (i.e. (10), (11) and (12))

$$ay'' + by' + cy = c_1\left(ar_1^2 + br_1 + c\right)e^{r_1 x} + c_2\left(ar_2^2 + br_2 + c\right)e^{r_2 x}. \quad (13)$$

The quantity in each of the parentheses on the right side of (13) is zero because r_1 and r_2 are roots of equation (9), therefore, y as given by (10) is indeed a general solution of (8). Now suppose that we want to find the **particular member** of the family of solutions also known as **particular solution** (10) that satisfies the initial conditions (6),

$$y(x_0) = y_0, \quad y'(x_0) = y_0'$$

By substituting $x = x_0$ and $y = y_0$ in (10) we obtain

$$c_1 e^{r_1 x_0} + c_2 e^{r_2 x_0} = y_0 \quad (14)$$

Similarly, setting $x = x_0$ and $y' = y_0'$ in (11) we obtain

$$c_1 r_1 e^{r_1 x_0} + c_2 r_2 e^{r_2 x_0} = y_0'. \quad (15)$$

On solving equations (14) and (15) simultaneously for c_1 and c_2, we find that

$$c_1 = \frac{y_0' - y_0 r_2}{r_1 - r_2} e^{-r_1 x_0}, \quad c_2 = \frac{y_0 r_1 - y_0'}{r_1 - r_2} e^{-r_2 x_0}. \quad (16)$$

Thus, no matter what initial conditions are assigned, that is, regardless of the values of x_0, y_0, and y_0' in equation (6), it is always possible to determine c_1 and c_2 so that the initial conditions are satisfied; moreover, there is only one possible choice of c_1 and c_2 for each set of initial conditions. With the values of c_1 and c_2 given by equation (16), the expression (10) is the solution of the initial value problem

$$ay'' + by' + cy = 0, \quad y(x_0) = y_0, \quad y'(x_0) = y_0'. \tag{17}$$

In general, consider the nth-order homogeneous linear differential equation

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = 0 \tag{18}$$

where $a_0, a_1, \cdots, a_{n-1}, a_n$ are real constants. The auxiliary or characteristic equation is given by

$$a_0 r^n + a_1 r^{n-1} + \cdots + a_{n-1} r + a_n = 0. \tag{19}$$

If the auxiliary equation (19) has the n distinct real roots $r_l, r_2, \cdots, r_n$, then the general solution of the nth-order ode (18) is

$$y = c_1 e^{r_1 x} + c_2 e^{r_2 x} + \cdots + c_n e^{r_n x} \tag{20}$$

where $c_1, c_2, \cdots, c_n$ are arbitrary constants.

Example

Find the general solution of the initial value problem

$$y'' + 5y' + 6y = 0, \quad y(0) = 2, \quad y'(0) = 3. \tag{i}$$

Solution. We assume that $y = e^{rx}$, then the first and second derivatives of y with respect to x are $y' = re^{rx}$, and $y'' = r^2 e^{rx}$ respectively. Putting y and its derivatives in (i) noting that $e^{rx} \neq 0$, to obtain the characteristic equation

$$r^2 + 5r + 6 = (r+2)(r+3) = 0.$$

The possible values of r are $r_1 = -2$ and $r_2 = -3$, the general solution of equation (i) is

$$y = c_1 e^{-2x} + c_2 e^{-3x}. \tag{ii}$$

Applying the first initial condition in (i), $y(0) = 2$ implies when $x = 0$ $y(0) = 2$, we have

$$c_1 + c_2 = 2. \tag{iii}$$

To use the second initial condition we must first differentiate equation (ii). This gives

$$y' = -2c_1 e^{-2x} - 3c_2 e^{-3x}.$$

Then, setting $x = 0$ and $y'(0) = 3$, we obtain

$$-2c_1 - 3c_2 = 3. \qquad \text{(iv)}$$

By solving equations (iii) and (iv) we find that $c_1 = 9$ and $c_2 = -7$. Using these values in the expression (ii), we obtain the solution

$$y = 9e^{-2x} - 7e^{-3x} \qquad \text{(v)}$$

of the initial value problem (i). The graph of the solution is shown in Figure 1.

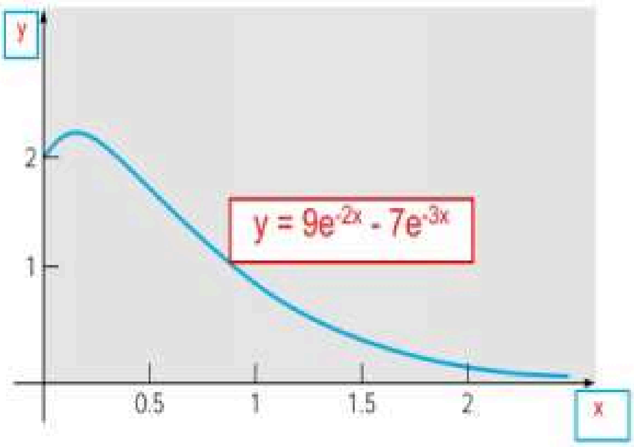

Figure: 1. Solution of $y'' + 5y' + 6y = 0, \quad y(0) = 2, \quad y'(0) = 3.$

Example

Find the particular solution of the initial value problem

$$4y'' - 8y' + 3y = 0, \quad y(0) = 2, \quad y'(0) = \frac{1}{2}. \tag{i}$$

Solution. If $y = e^{rx}$, $y' = re^{rx}$, $y'' = r^2 e^{rx}$, then the characteristic equation is

$$4r^2 - 8r + 3 = 0$$

and its roots are $r = r_1 = 3/2$ and $r = r_2 = 1/2$. Therefore the general solution of the differential equation is

$$y = c_1 e^{3x/2} + c_2 e^{x/2}. \tag{ii}$$

Using the initial conditions, we find that

$$c_1 + c_2 = 2, \qquad \frac{3}{2}c_1 + \frac{1}{2}c_2 = \frac{1}{2}.$$

The solution of these equations is $c_1 = -\frac{1}{2}$, $c_2 = \frac{5}{2}$, and the solution of the initial value problem (i) is

$$y(x) = -\frac{1}{2}e^{3x/2} + \frac{5}{2}e^{x/2}.$$

Figure 2 shows the graph of the solution.

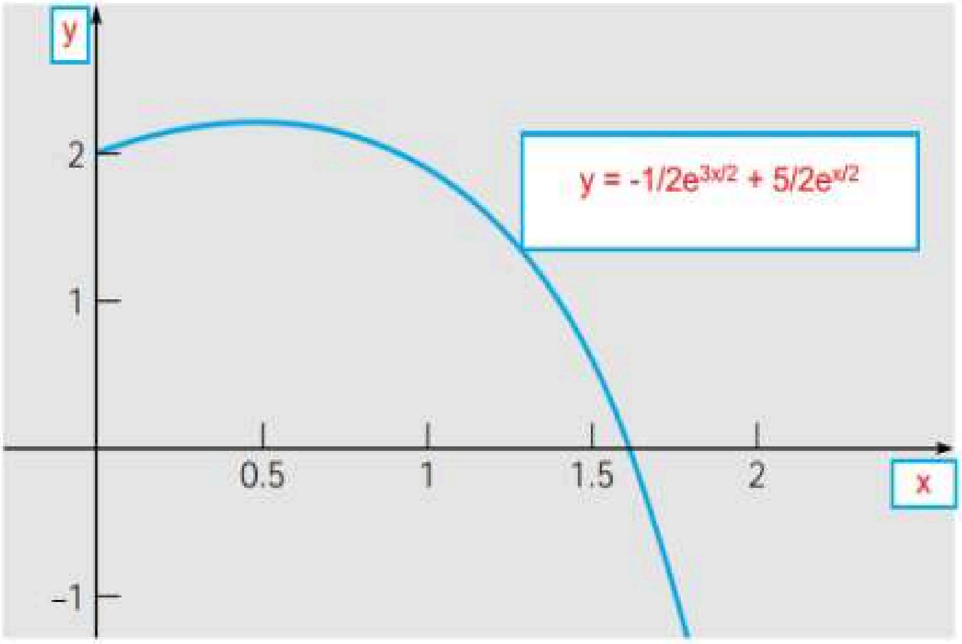

Figure: 2. Solution of $4y'' - 8y' + 3y = 0, \quad y(0) = 2, \quad y'(0) = \frac{1}{2}$

Exercises

In each of Exercises 1 through 10 find the general solution of the given differential equation, noting that $y = y(x)$.

1.	$y'' + 2y' - 3y = 0$	2.	$y'' + 3y' + 2y = 0$
3.	$6y'' - y' - y = 0$	4.	$2y'' - 3y' - y = 0$
5.	$y'' + 5y' = 0$	6.	$4y'' - 9y = 0$
7.	$y'' - 9y' + 9y = 0$	8.	$y'' - 2y' - 2y = 0$
9.	$y''' - 4y'' + y' + 6y = 0$	10.	$y'' - 3y' + 2y = 0$

In each of Problems 11 through 18 find the solution of the given initial value problem.

11.	$y'' + y' - 2y = 0,$	$y(0) = 1,$	$y'(0) = 1$
12.	$y'' + 4y' + 3y = 0,$	$y(0) = 2,$	$y'(0) = -1$
13.	$6y'' - 5y' + y = 0,$	$y(0) = 4,$	$y'(0) = 0$
14.	$y'' + 3y' = 0,$	$y(0) = -2,$	$y'(0) = 3$
15.	$y'' + 5y' + 3y = 0,$	$y(0) = 1,$	$y'(0) = 0$
16.	$2y'' + y' - 4y = 0,$	$y(0) = 0,$	$y'(0) = 1$
17.	$y'' + 8y' - 9y = 0,$	$y(1) = 1,$	$y'(1) = 0$
18.	$4y'' - y = 0,$	$y(-2) = 1,$	$y'(-2) = -1.$

Case 2. Repeated or Equal Roots

Consider again the ordinary differential equation (8) i.e.

$$ay'' + by' + cy = 0,$$

whose characteristic or auxiliary equation is given by (9) i.e.

$$ar^2 + br + c = 0.$$

Now we consider the second case, namely, that the two roots r_1 and r_2 are equal. This case occurs when the discriminant $b^2 - 4ac$ of (9) is zero, and it follows from the quadratic formula that

$$r = r_1 = r_2 = -\frac{b}{2a}.$$

The first independent solution of (8) in this case is

$$y_1 = y_1(x) = e^{-bx/2a}.$$

The second solution can be found using a method originated by D'Alembert. Recall that since $y_1(x)$ is a solution of (8), so is $cy_1(x)$ for any constant c.

By replacing c by a function $v = v(x)$ and then trying to determine v so that the product vy_1 is also a solution of (8). To see this, let

$$y = vy_1 \tag{21}$$

then the 1st and 2nd derivatives of y with respect to x are

$$\begin{aligned} y' &= v'y_1 + vy_1' \\ y'' &= v''y_1 + 2v'y_1' + vy_1''. \end{aligned}$$

Using y, y' and y'' in (8) and rearrange to get

$$av''y_1 + \big(2ay_1' + by_1\big)v' + \big(ay_1'' + by_1' + cy_1\big)v = 0. \tag{22}$$

From (22), since $y_1 = e^{-bx/2a}$ is a solution of (8), y_1 and its derivatives satisfy (8), i.e.

$$ay_1'' + by_1' + cy_1 = 0.$$

Next, use $y_1 = e^{-bx/2a}$ and its first derivative, the middle term of (22) yields is

$$\big(2ay_1' + by_1\big)v' = \left(2a\left(-\frac{b}{2a}\right) + b\right)e^{-bx/2a}v' = 0.$$

Equation (22) becomes

$$av''y_1 = av''e^{-bx/2a} = 0.$$

Since a is a positive constant and $e^{-bx/2a} \neq 0$ for all $x \in \mathbb{R}$, it follows that

$$\begin{aligned} v'' &= 0 \\ v' &= c_1 \\ v &= c_1 x + c_2, \end{aligned}$$

where c_1 and c_2 are arbitrary constants of integration. From the last equation and (21) we have

$$y = vy_1 = \left(c_1 x + c_2\right)e^{-bx/2a}.$$

Hence a two parameter family of solution (or general solution) of ode (8) in this case is given by

$$y(x) = \left(c_1 x + c_2\right)e^{rx}. \tag{23}$$

In general, we have the following about equal or repeated roots:

1. Consider the nth-order homogeneous linear differential equation (18) with constant coefficients. If the auxiliary equation (19) has the real root r occurring k times, then the part of the general solution of (18) corresponding to this k-fold repeated root is

$$\left(c_1 + c_2 x + c_3 x^2 + \cdots + c_k x^{k-1}\right) e^{rx}; \tag{24}$$

2. If,further, the remaining roots of the auxiliary equation (19) are the distinct real numbers $r_{k+1}, r_{k+2}, \cdots r_n$, then the general solution of (18) is

$$\begin{aligned} y(x) = &\left(c_1 + c_2 x + c_3 x^2 + \cdots + c_k x^{k-1}\right) e^{rx} \\ &+ c_{k+1} e^{r_{k+1} x} + c_{k+2} e^{r_{k+2} x} + \cdots + c_n e^{r_n x}; \end{aligned} \tag{25}$$

3. If, however, any of the remaining roots are also repeated, then the parts of the general solution of (18) corresponding to each of these other repeated roots are expressions similar to that corresponding to r, then the general solution of (18) is

$$\begin{aligned} y(x) = &\left(c_1 + c_2 x + c_3 x^2 + \cdots + c_k x^{k-1}\right) e^{rx} \\ &+ \left(c_{k+1} + c_{k+2} x + \cdots + c_n x^{n-1}\right) e^{r_1 x}. \end{aligned} \tag{26}$$

Example

Solve the differential equation

$$y'' + 4y' + 4y = 0. \tag{i}$$

The characteristic equation is

$$r^2 + 4r + 4 = (r+2)^2 = 0,$$

so $r = r_1 = r_2 = -2$. Therefore, the general solution (i), using equation (23) is

$$y(x) = \left(c_1 + c_2 x\right) e^{-2x}.$$

Example

Find the solution of the initial value problem

$$y'' - y' + 0.25y = 0, \quad y(0) = 2, \quad y'(0) = \frac{1}{3}. \tag{i}$$

The characteristic equation is

$$r^2 - r + 0.25 = 0,$$

the roots are $r = r_1 = r_2 = \frac{1}{2}$.

The general solution of the differential equation (i) is

$$y(x) = \big(c_1 + c_2 x\big)e^{x/2}. \tag{ii}$$

Applying the first initial condition (i.e. $y(0) = 2$) in (ii) we find that

$$y(0) = c_1 = 2.$$

To satisfy the second initial condition (i.e. $y'(0) = \frac{1}{3}$), we first differentiate equation (ii) and then set $x = 0$. This gives

$$y'(0) = \frac{1}{2}c_1 + c_2 = \frac{1}{3},$$

so that $c_2 = -2/3$. Thus, the solution of the initial value problem (i) is

$$y(x) = \big(2 - \frac{2}{3}x\big)e^{x/2}.$$

Exercises

In each of Problems 1 through 10 find the general solution of the given differential equation.

1.	$y'' - 2y' + y = 0$	2.	$9y'' + 6y' + y = 0$
3.	$4y'' - 4y' - 3y = 0$	4.	$4y'' + 12y' + 9y = 0$
5.	$y'' - 2y' + 10y = 0$	6.	$y'' - 6y' + 9y = 0$
7.	$4y'' + 17y' + 4y = 0$	8.	$16y'' + 24y' + 9y = 0$
9.	$25y'' - 20y' + 4y = 0$	10.	$2y'' + 2y' + y = 0$

In each of Problems 11 through 15 solve the given initial value problem. Sketch the graph of the solution and describe its behaviour for increasing x.

11.	$9y'' - 12y' + 4y = 0,$	$y(0) = 2,$	$y'(0) = -1$
12.	$y'' - 6y' + 9y = 0,$	$y(0) = 0,$	$y'(0) = 2$
13.	$9y'' + 6y' + 82y = 0,$	$y(0) = -1,$	$y'(0) = 2$
14.	$4y'' + 12y' + 9y = 0,$	$y(0) = 1,$	$y'(0) = -4$
15.	$y'' + 4y' + 4y = 0,$	$y(-1) = 2,$	$y'(-1) = 1$

16. Find the general solution of each of the following odes

$$y''' - 4y'' - 3y' + 18y = 0 \text{ and } y^{(iv)} - 5y''' + 6y'' + 4y' - 8y = 0.$$

Case 3. Complex Roots of the Characteristic Equation.

We continue with the ordinary differential equation (8) i.e.

$$ay'' + by' + cy = 0,$$

whose characteristic or auxiliary equation is given by (9) i.e.

$$ar^2 + br + c = 0.$$

Suppose now that $b^2 - 4ac$ is negative. Then the roots of (9) are conjugate complex numbers, we denote them by

$$r_1 = \alpha + i\beta, \qquad r_2 = \alpha - i\beta, \tag{27}$$

where α and β are real. The corresponding part of the general solution is

$$k_1 e^{(\alpha+i\beta)x} + k_2 e^{(\alpha-i\beta)x},$$

where k_1 and k_2 are constants. The solutions defined by $e^{(\alpha+i\beta)x}$ and $e^{(\alpha-i\beta)x}$ are complex function of real variable x. Using Euler's formula

$$e^{i\theta} = \cos\theta + i\sin\theta$$

which holds for all real θ, we have

$$
\begin{aligned}
k_1 e^{(\alpha+i\beta)x} + k_2 e^{(\alpha-i\beta)x} &= k_1 e^{\alpha x} e^{i\beta x} + k_2 e^{\alpha x} e^{-i\beta x} \\
&= \big[k_1\big(\cos\beta x + i\sin\beta x\big) + k_2\big(\cos\beta x - i\sin\beta x\big)\big] e^{\alpha x} \\
&= \big[(k_1+k_2)\cos\beta x + i(k_1-k_2)\sin\beta x\big] e^{\alpha x} \\
&= \big[c_1\cos\beta x + c_2\sin\beta x\big] e^{\alpha x}
\end{aligned}
$$

where $c_1 = k_1 + k_2$ and $c_2 = i(k_1 - k_2)$ are two new arbitrary constants. Thus a two parameters family of solution of (8) to the non repeated complex root $\alpha \pm \beta$ of (9) is

$$
y(x) = \big[c_1\cos\beta x + c_2\sin\beta x\big] e^{\alpha x} \tag{28}
$$

Example

Find the general solution of

$$
y'' + y' + y = 0 \tag{i}
$$

The characteristic equation is

$$
r^2 + r + 1 = 0,
$$

and its roots are

$$r = \frac{-1 \pm \sqrt{3}}{2} = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}.$$

Thus $\alpha = -1/2$ and $\beta = \sqrt{3}/2$ now use equation (28) to obtain the two parameter family of solution of (i) as

$$y(x) = \left[c_1 \cos\left(\sqrt{3}x/2\right) + c_2 \sin\left(\sqrt{3}x/2\right)\right] e^{-x/2}$$

Example

Find the solution of the initial value problem

$$16y'' - 8y' + 145y = 0, \quad y(0) = -2, \quad y'(0) = 1. \tag{i}$$

The characteristic equation is

$$16r^2 - 8r + 145 = 0$$

and its roots are

$$r = 1/4 \pm 3i.$$

Thus the general solution of the differential equation is

$$y(x) = \left[c_1 \cos 3x + c_2 \sin 3x\right] e^{x/4}. \tag{ii}$$

Use the first initial condition, setting $x = 0$ in equation (ii), this gives $c_1 = -2$. For the second initial condition, we must differentiate (ii) and then set $x = 0$, to get

$$\frac{1}{4}c_1 + 3c_2 = 1.$$

Now use $-2 = c_1$ in the last equation to get $c_2 = 1/2$. Using these values of c_1 and c_2 in (ii), we obtain the particular or unique solution of (i) to be

$$y(x) = \left[-2\cos 3x + \frac{1}{2}\sin 3x \right] e^{x/4}.$$

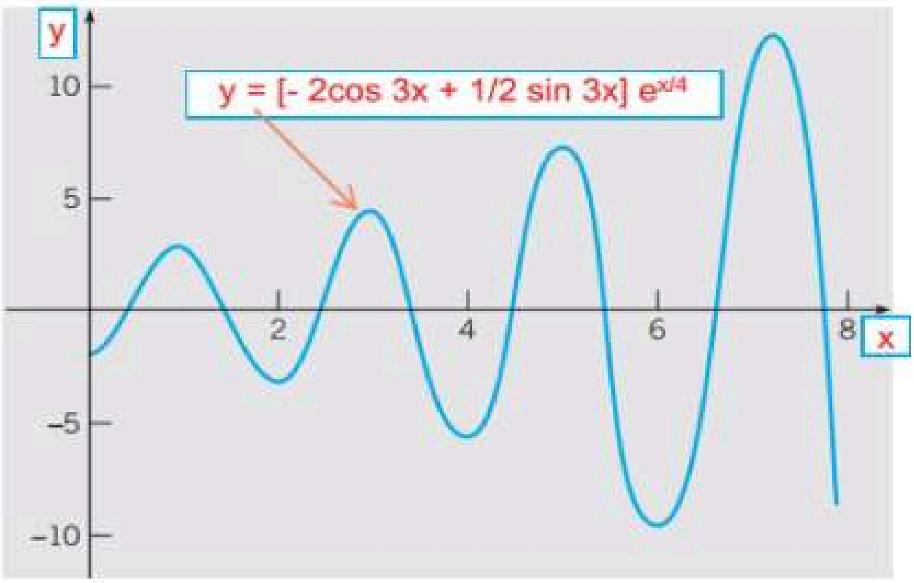

Figure: 3. Solution of $16y'' - 8y' + 145y = 0, \quad y(0) = -2, \quad y'(0) = 1.$

Exercises

In each of Problems 1 through 6 use Euler's formula to write the given expression in the form $a + ib$.

1.	$\exp(1+2i)$	2.	$\exp(2-3i)$	3.	$e^{i\pi}$
4.	$e^{2-(\pi/2)i}$	5.	2^{1-i}	6.	π^{-1+2i}

In each of Problems 7 through 16 find the general solution of the given differential equation.

7.	$y'' - 2y' + 2y = 0$	8.	$y'' - 2y' + 6y = 0$
9.	$y'' + 2y' - 8y = 0$	10.	$y'' + 2y' + 2y = 0$
11.	$y'' + 6y' + 13y = 0$	12.	$4y'' + 9y = 0$
13.	$y'' + 2y' + 1.25y = 0$	14.	$9y'' + 9y' - 4y = 0$
15.	$y'' + y' + 1.25y = 0$	16.	$y'' + 4y' + 6.25y = 0$

In each of Problems 17 through 22 find the solution of the given initial value problem.

17.	$y'' + 4y = 0,$	$y(0) = 0,$	$y'(0) = 1$
18.	$y'' + 4y' + 5y = 0,$	$y(0) = 1,$	$y'(0) = 0$
19.	$y'' - 2y' + 5y = 0,$	$y(\pi/2) = 0,$	$y'(\pi/2) = 2$
20.	$y'' + y = 0,$	$y(\pi/3) = 2,$	$y'(\pi/3) = -4$
21.	$y'' + y' + 1.25y = 0,$	$y(0) = 3,$	$y'(0) = 1$
22.	$y'' + 2y' + 2y = 0,$	$y(\pi/4) = 2,$	$y'(\pi/4) = -2$

References

1. Robert Alexander Frazer, William Jolly Duncan, Arthur Roderich Collar - Elementary Matrices And Some Applications To Dynamics And Differential Equations - Cambridge University Press-(1947)
2. Haberman R. - Elementary Applied Partial Differential Equations with Fourier Series and Boundary Value Problems-(1987).
3. C. Henry Edwards, David E. Penney - Elementary Differential Equations-Prentice Hall (2007).
4. Henry Thomas Herbert Piaggio - An elementary treatise on differential equations and their applications- G. Bell and Sons, LTD. London (1929).
5. Morris Tenenbaum, Harry Pollard - Ordinary differential equations: an elementary textbook for students of mathematics, engineering, and the sciences-Dover Publications, (1985).
6. Patrick J. Murphy - Elementary Matrices and Some Applications to Dynamics and Differential Equations-Cambridge University Press (1960).
7. Shepley L. Ross - Introduction to Ordinary Differential Equations, 4th Edition-Wiley (1989).
8. William E. Boyce, Richard C. DiPrima - Elementary Differential Equations And Boundary Value Problems-Wiley (2000).

MTH 201
Differential Equation III

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Nonhomogeneous Equations

Consider the (nonhomogeneous) differential equation

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = F(x), \tag{1}$$

where the coefficients $a_0, a_1, \cdots, a_{n-1}, a_n$ are constants, with $a_0 \neq 0$, and the nonhomogeneous term $F = F(x)$ is (in general) a **nonconstant function** of x. Recall that the general solution of (1) may be written

$$y(x) = y_c + y_p \tag{2}$$

where $y_c = y_c(x)$ is the **complementary function**, that is, the general solution of the corresponding homogeneous equation (Equation (1) with F replaced by 0), and $y_p = y_p(x)$ is a **particular integral**, that is, any solution of (1) containing no arbitrary constants. In Module 8.3 we learned how to find the **complementary function** or **solution of homogeneous equation,** now we consider methods of determining a particular integral. We consider first the method of undetermined coefficients. Mathematically speaking, the class of functions F to which this method applies is actually quite restricted; but this mathematically narrow class includes functions of frequent occurrence and considerable importance in various physical applications.

Method of Undetermined Coefficients

Example

Determine the particular integral for the ordinary differential equation

$$y'' - 2y' - 3y = 2e^{4x}. \tag{i}$$

We proceed to seek a particular integral y_p. The differential equation (i) requires a solution y_p which is such that its second derivative i.e. y_p'', minus twice its first derivative $(-2y_p')$, minus three times the solution itself $(-3y_p)$, add up to twice the exponential function e^{4x} i.e.

$$y_p'' - 2y_p' - 3y_p = 2e^{4x}.$$

Since the derivatives of e^{4x} are constant multiples of e^{4x}, it seems reasonable that the desired particular integral might also be a constant multiple of e^{4x}. Thus we assume a particular integral of the form

$$y_p = Ae^{4x}, \tag{ii}$$

where A is a constant (undetermined coefficient) to be determined such that (ii) is a solution of (i). Differentiating (ii), we obtain

Then substituting y_p, y_p' and y_p'' into (i), we obtain

$$16Ae^{4x} - 2(4Ae^{4x}) - 3(Ae^{4x}) = 2e^{4x} \quad \text{or} \quad 5Ae^{4x} = 2e^{4x} \ \Rightarrow \ A = \frac{2}{5}.$$

Thus the particular integral (particular solution) becomes

$$y_p = \frac{2}{5}e^{4x}.$$

Example

Find a particular solution of

$$y'' - 3y' - 4y = 2e^{-x}. \tag{iii}$$

Proceeding as in the above example, we assume that $y_p = Ae^{-x}$. By substituting y_p, y_p' and y_p'' in (iii), we obtain

$$(A + 3A - 4A)e^{-x} = 2e^{-x} \quad \text{or} \quad 0 = 2e^{-x}. \tag{iv}$$

Since the left side of Eq. (iv) is zero, there is no choice of A that satisfies this equation. Therefore, there is no particular solution of Eq. (iii) of the assumed form.

The reason for this possibly unexpected result becomes clear if we solve the homogeneous equation

$$y'' - 3y' - 4y = 0 \tag{v}$$

that corresponds to Eq. (iii). A fundamental set of solutions of Eq. (v) are $y_1(x) = e^{-x}$ and $y_2(x) = e^{4x}$. Thus our assumed particular solution of Eq. (iii) is actually a solution of the homogeneous equation (v); consequently, it cannot possibly be a solution of the nonhomogeneous equation (iii). To find a solution of Eq. (iii), we must therefore consider functions different from the complementary function. In this case we assume

$$y_p = Axe^{-x}, \quad y_p' = -Axe^{-x} + Ae^{-x}, \quad y_p'' = Axe^{-x} - 2Ae^{-x}.$$

Substituting y_p and its derivatives into Eq. (iii), we find that

$$-5A = 2 \quad \Rightarrow \quad A = -\frac{2}{5}.$$

The particular integral for Eq. (iii) is

$$y_p = -\frac{2}{5}xe^{-x}.$$

Example

Find the general solution of the ordinary differential equation

$$y'' - 2y' - 3y = 2e^{3x} \tag{vi}$$

.Consider first, the homogeneous equation

$$y'' - 2y' - 3y = 0.$$

Let $y = e^{rx}$ then $y' = re^{rx}$ and $y'' = r^2e^{rx}$. The characteristic equation is

$$r^2 - 2r - 3 = (r-3)(r+1) = 0,$$

with roots $r_1 = 3$ and $r_2 = -1$. The complementary function is

$$y_c = c_1e^{3x} + c_2e^{-x}. \tag{vii}$$

Since e^{3x} is in the complementary function, let the particular integral be

$$y_p = Axe^{3x}, \tag{viii}$$

then

$$y_p' = 3Axe^{3x} + Ae^{3x} \quad \text{and} \quad y_p'' = 9Axe^{3x} + 6Ae^{3x}.$$

Substituting y_p and its derivatives in Eq. (vi) we find that $A = 1/2$. Eq. (viii) becomes

$$y_p = \frac{1}{2}xe^{3x}, \qquad \text{(ix)}.$$

From equations (vii) and (ix), using Eq. (2), the required general solution is

$$y(x) = y_c + y_p = c_1 e^{3x} + c_2 e^{-x} + \frac{1}{2}xe^{3x}.$$

The Method of Undetermined Coefficients(UC)

Definition

We shall call a function a UC function if it is either

1. a function defined by one of the following:
   - (i) x^n, where n is a positive integer or zero;
   - (ii) e^{ax}, where a is a constant $\neq 0$,
   - (iii) $\sin(bx + c)$, where b and c are constants, $b \neq 0$;
   - (iv) $\cos(bx + c)$, where b and c are constants, $b \neq 0$;

   or
2. a function defined as a finite product of two or more functions of these four types.

Example

Examples of UC functions of the four basic types (i), (ii), (iii), (iv) of the preceding definition are those defined, respectively, by

$$x^5, \quad e^{-3x}, \quad \sin(3x/4), \quad \cos(2x+\pi/4).$$

Examples of UC functions defined as finite products of two or more of these four basic types are those defined, respectively, by

$$x^3 e^{4x}, \quad \sin 2x \cos 5x, \quad x \sin 2x, \quad e^{-2x} \sin 7x, \quad x^6 e^{-5x} \sin 6x.$$

Definition

Consider a *UC* function f. The set of functions consisting of f itself and all linearly independent *UC* functions of which the successive derivatives of f are either constant multiples or linear combinations will be called the *UC* set of f.

Example

The function f defined for all real x by $f(x) = x^4$ is a *UC* function. Computing derivatives of f, we find

$$f'(x) = 4x^3, \quad f''(x) = 12x^2, \quad f'''(x) = 24x,$$
$$f^{(iv)}(x) = 24 \times 1, \quad f^{(n)}(x) = 0 \text{ for } n > 4.$$

The linearly independent *UC* functions of which the successive derivatives of f are either constant multiples or linear combinations are those given by

$$x^3, \quad x^2, \quad x, \quad 1.$$

Thus the *UC* set of x^4 is the set

$$S = \{x^4, \quad x^3, \quad x^2, \quad x, \quad 1\}.$$

Example

The function f defined for all real x by $f(x) = \sin 2x$ is a *UC* function. Computing derivatives of f, we find

$$f'(x) = 2\cos 2x, \quad f''(x) = -4\sin 2x,$$

The only linearly independent UC function of which the successive derivatives of f are constant multiples or linear combinations is that given by $\cos 2x$. Thus the UC set of $\sin 2x$ is the set

$$S = \{\sin 2x, \cos 2x\}.$$

Example

Find the general solution of the ode

$$y'' - 2y' - 3y = 2e^{x} - 10\sin x. \quad \text{(i)}$$

The corresponding homogeneous equation is

$$y'' - 2y' - 3y = 0. \quad \text{(ii)}$$

and the complementary function of Eq. (ii) is

$$y_c = c_1 e^{3x} + c_2 e^{-x}. \quad \text{(iii)}$$

The nonhomogeneous term is the linear combination $2e^{x} - 10\sin x$ of the two UC functions given by e^{x} and $\sin x$.

1. Form the UC set for each of these two functions. We find

$$S_1 = \{e^x\}, \quad S_2 = \{\sin x, \cos x\}.$$

2. Note that neither of these sets is identical with nor included in the other; hence both are retained.
3. Examining the complementary function, we see that none of the functions $e^x, \sin x, \cos x$ in either of these sets is a solution of the corresponding homogeneous equation. Hence neither set needs to be revised.
4. Thus the original sets S_1 and S_2 remain intact in this problem, and we form the linear combination

$$Ae^x + B\sin x + C\cos x$$

of the three elements $e^x, \sin x, \cos x$ of S_1 and S_2, with the undetermined coefficients A, B, C.
5. We determine these unknown coefficients by substituting the linear combination formed in Step 4 and its derivatives into the differential equation and demanding that it satisfy the differential equation identically. That is, we take

$$y_p = Ae^x + B\sin x + C\cos x, \quad \text{(iv)}$$

as particular solution. Then

$$y_p' = Ae^x + B\cos x - C\sin x, \quad y_p'' = Ae^x - B\sin x - C\cos x.$$

Substituting y_p and its derivatives in (i) and simply to get

$$-4Ae^x + (-4B + 2C)\sin x + (-4C - 2B)\cos x = 2e^x - 10\sin x.$$

Equating coefficients of these like terms, we obtain the equations

$$-4A = 2, \quad -4B + 2C = -10, \quad -4C - 2B = 0.$$

From these equations, we find that

$$A = -\frac{1}{2}, \quad B = 2, \quad C = -1,$$

and hence we obtain the particular integral

$$y_p = -\frac{1}{2}e^x + 2\sin x - \cos x, \tag{v}$$

Thus from equations (iii) and (v) the general solution of the differential equation (i) is

$$y(x) = y_c + y_p = c_1 e^{3x} + c_2 e^{-x} - \frac{1}{2}e^x + 2\sin x - \cos x.$$

Example

Find the general solution of the ordinary differential equation

$$y'' - 3y' + 2y = 2x^2 + e^x + 2xe^x + 4e^{3x}. \tag{i}$$

The corresponding homogeneous equation is

$$y'' - 3y' + 2y = 0 \tag{ii}$$

and the complementary function is

$$y_c = c_1 e^x + c_2 e^{2x} \tag{iii}$$

The nonhomogeneous term is the linear combination

$$2x^2 + e^x + 2xe^x + 4e^{3x}$$

of the four DC functions given by x^2, e^x, xe^x and e^{3x}

1. Form the UC set for each of these functions. We have

$$S_1 = \{x^2, x, 1\}, \quad S_2 = \{e^x\}, \quad S_3 = \{xe^x, e^x\}, \quad S_4 = \{e^{3x}\}.$$

2. We note that S_2 is completely included in S_3, so S_2 is omitted from further consideration, leaving the three sets

$$S_1 = \{x^2, x, 1\}, \quad S_3 = \{xe^x, e^x\}, \quad S_4 = \{e^{3x}\}.$$

3. We also observe that $S_3 = \{xe^x, e^x\}$ includes e^x, which is included in the complementary function and so is a solution of the corresponding homogeneous differential equation. Thus we multiply each member of S_3 by x to obtain the revised family

$$S_3' = \{x^2 e^x, xe^x\}$$

which contains no members that are solutions of the corresponding homogeneous equation.

4. Thus there remain the original UC sets

$$S_1 = \{x^2, x, 1\}, \qquad S_4 = \{e^{3x}\}, \quad S_3' = \{x^2 e^x, xe^x\}.$$

These contain the six elements

$$x^2, x, 1, e^{3x}, x^2 e^x, xe^x$$

We form the linear combination

$$Ax^2 + Bx + C + De^{3x} + Ex^2 e^x + Fxe^x,$$

of these six elements.

5. Let the particular integral be

$$y_p = Ax^2 + Bx + C + De^{3x} + Ex^2 e^x + Fxe^x, \qquad \text{(iv)}$$

find the 1st and 2nd derivatives of (iv) with respect to x, substituting in Eq. (i) and simply to obtain

$$(2A - 3B + 2C) + (2B - 6A)x + 2Ax^2 + 2De^{3x} - 2Exe^x$$
$$+ (2E - F)e^x = 2x^2 + e^x + 2xe^x + 4e^{3x}.$$

Equating coefficients of like terms, results

$$\begin{aligned} 2A - 3B + 2C &= 0, \\ 2B - 6A &= 0, \\ 2A &= 2, \\ 2D &= 4, \\ -2E &= 2, \\ 2E - F &= 1. \end{aligned}$$

Solve these equations to get

$$A = 1,\ B = 3,\ C = \frac{7}{2},\ D = 2,\ E = -1,\ F = -3,$$

equation (iv) becomes

$$y_p = x^2 + 3x + \frac{7}{2} + 2e^{3x} - x^2 e^x - 3xe^x. \qquad \text{(v)}$$

From (iii) and (v) the required general solution of ode (i) is

$$y(x) = c_1 e^x + c_2 e^{2x} + x^2 + 3x + \frac{7}{2} + 2e^{3x} - x^2 e^x - 3xe^x.$$

Exercises

Find the general solution of each of the ordinary differential equations in Exercises 1-10.

1. $y'' - 3y' + 2y = 4x^2$.
2. $y'' - 2y' - 8y = 4e^{2x} - 21e^{-3x}$.
3. $y'' + 2y' + 5y = 6\sin 2x + 7\cos 2x$.
4. $y'' + 2y' + 2y = 10\sin 4x$.
5. $y'' + 2y' + 4y = 13cos4x$.
6. $y'' - 3y' - 4y = 16x - 12e^{2x}$.
7. $y'' + 6y' + 5y = 2e^x + 10e^{5x}$.
8. $y'' + 2y' + 10y = 5xe^{-2x}$.
9. $2y'' + 3y' - 2y = 6x^2 e^x - 4x^2 + 12$.
10. $y'' + 6y' + 8y = 6xe^{2x} + 8x^2$.

These and similar examples of the four basic types of UC functions

SN	UC Function	UC Set
1.	x^n	$\{x^n, x^{n-1}, x^{n-2}, \cdots, x, 1\}$
2.	e^{ax}	$\{e^{ax}\}$
3.	$sin(bx+c)$ *or* $cos(bx+c)$	$\{\sin(bx+c), \cos(bx+c)\}$
4.	$x^n e^{ax}$	$\{x^n e^{ax}, x^{n-1}e^{ax}, x^{n-2}e^{ax}, \cdots, xe^{ax}, e^{ax}\}$
5.	$x^n \sin(bx+c)$ *or* $x^n \cos(bx+c)$	$\{x^n \sin(bx+c), x^n \cos(bx+c),$ $x^{n-1}\sin(bx+c), x^{n-1}\cos(bx+c), \cdots,$ $x\sin(bx+c), x\cos(bx+c),$ $\sin(bx+c), \cos(bx+c)\}$
6.	$e^{ax}\sin(bx+c)$ *or* $e^{ax}\cos(bx+c)$	$\{e^{ax}\sin(bx+c), e^{ax}\cos(bx+c)\}$
7.	$x^n e^{ax}\sin(bx+c)$ *or* $x^n e^{ax}\cos(bx+c)$	$\{x^n e^{ax}\sin(bx+c), x^n e^{ax}\cos(bx+c),$ $x^{n-1}e^{ax}\sin(bx+c), x^{n-1}e^{ax}\cos(bx+c), \cdots,$ $xe^{ax}\sin(bx+c), xe^{ax}\cos(bx+c),$ $e^{ax}\sin(bx+c), e^{ax}\cos(bx+c), \}$

Variation of Parameters

While the process of carrying out the method of undetermined coefficients is actually quite straightforward, the method applies in general to a rather small class of problems. For example, it would not apply to the apparently simple equation

$$y'' + y = \tan x.$$

Observe that this problem does not fall within the scope of the method of undetermined coefficients because the nonhomogeneous term $F(x) = \tan x$ involves a quotient (rather than a sum or a product) of $\sin x$ or $\cos x$. We thus seek a method of finding a particular integral that applies in all cases (including variable coefficients) in which the complementary function is known. Such a method is **the method of variation of parameters.** We shall develop this method in connection with the general second-order linear differential equation with variable coefficients

$$a_0(x)y'' + a_1(x)y' + a_2(x)y = F(x). \tag{3}$$

Suppose that $y_1 = y_1(x)$ and $y_2 = y_2(x)$ are linearly independent solutions of the corresponding homogeneous equation

$$a_0(x)y'' + a_1(x)y' + a_2(x)y = 0. \tag{4}$$

Then the complementary function of Equation (3) is

$$c_1 y_1 + c_2 y_2,$$

where y_1 and y_2 are linearly independent solutions of (4), and c_1 and c_2 are arbitrary constants. The procedure in the method of variation of parameters is to replace the arbitrary constants c_1 and c_2 in the complementary function by respective functions $v_1 = v_1(x)$ and $v_2 = v_2(x)$ which will be determined so that the resulting function, which is defined by

$$v_1 y_1 + v_2 y_2, \tag{5}$$

will be a particular integral of Equation (3) (hence the name, variation of parameters). We assume a solution of the form (5) and write

$$y_p = y_p(x) = v_1 y_1 + v_2 y_2, \tag{6}$$

Differentiating (6), we find

$$y_p' = v_1' y_1 + v_1 y_1' + v_2' y_2 + v_2 y_2'. \tag{7}$$

At this point we impose the condition

$$v_1' y_1 + v_2' y_2 = 0. \tag{8}$$

Eq. (7) becomes

$$y_p' = v_1 y_1' + v_2 y_2'. \tag{9}$$

Differentiating (9) to obtain

$$y_p' = v_1' y_1' + v_1 y_1'' + v_2' y_2' + v_2 y_2''. \tag{10}$$

Thus we substitute (6), (9), and (10) for y, y', and y'', respectively, in Eq. (3) and simplify to obtain

$$\begin{aligned} &v_1[a_0(x)y_1'' + a_1(x)y_1' + a_2(x)y_1] \\ &+ v_2[a_0(x)y_2'' + a_1(x)y_2' + a_2(x)y_2] \\ &+ a_0(x)[v_1'y_1' + v_2'y_2'] = F(x). \end{aligned} \tag{11}$$

Since y_1 and y_2 are solutions of the corresponding homogeneous differential equation (4), the expressions in the first two brackets in (11) are identically zero. This leaves

$$v_1'y_1' + v_2'y_2' = \frac{F(x)}{a_0(x)}. \tag{12}$$

Thus the two imposed conditions require that the functions v_1 and v_2 be chosen such that the system of equations

$$
\begin{aligned}
v_1' y_1 + v_2' y_2 &= 0, \\
v_1' y_1' + v_2' y_2' &= \frac{F(x)}{a_0(x)},
\end{aligned}
\tag{13}
$$

is satisfied. The determination of coefficients of this system is

$$
W[y_1(x), y_2(x)] = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}
$$

Since y_1 and y_2 are linearly independent solutions of the corresponding homogeneous differential equation (4), we know that $W[y_1(x), y_2(x)] \neq 0$. Hence the system (13) has a unique solution. Actually solving this system, we obtain

$$
v_1' = \frac{\begin{vmatrix} 0 & y_2 \\ \frac{F(x)}{a_0(x)} & y_2' \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} = -\frac{F(x) y_2}{a_0(x) W[y_1(x), y_2(x)]}
$$

$$v_2' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & \frac{F(x)}{a_0(x)} \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} = \frac{F(x)y_1}{a_0(x)W[y_1(x), y_2(x)]},$$

Thus we obtain the functions v_1 and v_2 defined by

$$\begin{aligned} v_1 &= -\int^x \frac{F(s)y_2(s)}{a_0(s)W[y_1(s), y_2(s)]} ds \\ v_2 &= \int^x \frac{F(s)y_1(s)}{a_0(s)W[y_1(s), y_2(s)]} ds. \end{aligned} \tag{14}$$

Therefore a particular integral y_p of Eq. (3) is defined by Eq. (6) i. e.

$$y_p = v_1 y_1 + v_2 y_2,$$

where v_1 and v_2 are defined by (14).

Example

Consider the differential equation

$$y'' + y = \tan x. \tag{i}$$

The solution of the homogeneous (or complementary function) associated with (i) is defined by

$$y_c(x) = c_1 \sin x + c_2 \cos x. \tag{ii}$$

Let the particular integral of the ode (i) be defined as

$$y_p = v_1 \sin x + v_2 \cos x, \tag{iii}$$

where the functions v_1 and v_2 will be determined. Then

$$y_p' = v_1' \sin x + v_1 \cos x + v_2' \cos x - v_2 \sin x, \tag{iv}$$

Impose the condition

$$v_1' \sin x + v_2' \cos x = 0, \tag{v}$$

so that Eq (iv) becomes

$$y_p' = v_1 \cos x - v_2 \sin x. \tag{vi}$$

From (vi) we have

$$y_p'' = v_1' \cos x - v_1 \sin x - v_2' \sin x - v_2 \cos x. \tag{vii}$$

Substituting (iii) and (vii) into (i) to get

$$v_1' \cos x - v_2' \sin x = \tan x. \tag{viii}$$

Thus we have the two equations (v) and (viii) from which to determine v_1' and v_2'

$$v_1' \sin x + v_2' \cos x = 0$$
$$v_1' \cos x - v_2' \sin x = \tan x.$$

Solving these equations

$$v_1' = \frac{\begin{vmatrix} 0 & \cos x \\ \tan x & -\sin x \end{vmatrix}}{\begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix}} = \frac{-\cos x \tan x}{-1} = \sin x$$

$$v_2' = \frac{\begin{vmatrix} \sin x & 0 \\ \cos x & \tan x \end{vmatrix}}{\begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix}} = \frac{\sin x \tan x}{-1} = \frac{\sin^2 x}{\cos x}$$
$$= \frac{\cos^2 x - 1}{\cos x} = \cos x - \sec x.$$

Integrating results

$$v_1 = -\cos x + c_3 \quad \text{and} \quad v_2 = \sin x - \ln|\sec x + \tan x| + c_4. \qquad \text{(ix)}$$

Substituting Eq. (ix) in Eq. (iii) we obtain

$$y_p = c_3 \sin x + c_4 \cos x - \cos x(\ln|\sec x + \tan x|)$$

Since a particular integral is a solution free of arbitrary constants, we may assign any particular values A and B to c_3 and c_4, respectively, and the result will be the particular integral

$$y_p = A \sin x + B \cos x - \cos x(\ln|\sec x + \tan x|)$$

Thus the general solution

$$\begin{aligned} y(x) &= y_c + y_p \\ &= (c_1 + A)\sin x + (c_2 + B)\cos x - \cos x(\ln|\sec x + \tan x|) \\ &= k_1 \sin x + k_2 \cos x - \cos x(\ln|\sec x + \tan x|) \end{aligned}$$

where $k_1 = (c_1 + A)$ and $k_2 = (c_2 + B)$.

Example

Find a particular solution of

$$y'' + 4y = 3\csc t. \tag{i}$$

Note that in this example $y = y(t)$.

The complementary function is

$$y_c = c_1 \cos 2t + c_2 \sin 2t \tag{ii}$$

Let the particular integral be defined as

$$y_p = v_1 \cos 2t + v_2 \sin 2t, \tag{iii}$$

where $v_1 = v_1(t)$ and $v_2 = v_2(t)$. Differentiating equation (iii) with respect to t, results

$$y_p' = v_1' \cos 2t - 2v_1 \sin 2t + v_2' \sin 2t + 2v_2 \cos 2t. \tag{iv}$$

Impose the usual condition

$$v_1' \cos 2t + v_2' \sin 2t = 0, \tag{v}$$

equation (iii) reduces to

$$y_p' = -2v_1 \sin 2t + 2v_2 \cos 2t. \tag{vi}$$

Next, differentiating (v), we obtain

$$y_p'' = -2v_1' \sin 2t - 4\,v_1 \cos 2t + 2v_2' \cos 2t - 4\,v_2 \sin 2t. \tag{vii}$$

Substituting equation (iii) and (vi) in (i) and simplify to get

$$-2v_1' \sin 2t + 2v_2' \cos 2t = 3 \csc t. \qquad \text{(viii)}$$

From equations (v) and (viii) we have

$$\begin{aligned} v_1' \cos 2t + v_2' \sin 2t &= 0 \\ -2v_1' \sin 2t + 2v_2' \cos 2t &= 3 \csc t \end{aligned}$$

Solving these equations, we find that

$$\begin{aligned} v_1' &= -3 \cos t \\ v_2' &= \frac{3}{2} \csc t - 3 \sin t. \end{aligned}$$

Integrating to obtain

$$v_1 = -3 \sin t + c_3 \quad \text{and} \quad v_2 = \frac{3}{2} \ln|\csc t - \cot t| + 3 \cos t + c_4.$$

The particular integral is

$$y_p = [-3\sin t + c_3]\cos 2t + \left[\frac{3}{2}\ln|\csc t - \cot t| + 3\cos t + c_4\right]\sin 2t.$$

The general solution of equation (i) is

$$\begin{aligned} y(t) &= y_c + y_p \\ &= [c_1 + c_3]\cos 2t + [c_2 + c_4]\sin 2t \\ &+ \frac{3}{2}\ln|\csc t - \cot t|\sin 2t + 3[\sin 2t\cos t - \cos 2t\sin t] \\ &= k_1\cos 2t + k_2\sin 2t + \frac{3}{2}\ln|\csc t - \cot t|\sin 2t + 3\sin(2t - t) \\ &= k_1\cos 2t + k_2\sin 2t + \frac{3}{2}\ln|\csc t - \cot t|\sin 2t + 3\sin t, \end{aligned}$$

where $k_1 = c_1 + c_3$ and $k_2 = c_2 + c_4$. The terms involving the arbitrary constants k_1 and k_2 are the general solution of the corresponding homogeneous equation, while the remaining terms are a particular solution of the nonhomogeneous equation (i). Therefore, the last is the general solution of Eq. (i).

Exercises

In each of Exercises 1 through 4 use the method of variation of parameters to find a particular solution of the given ordinary differential equations. Then check your answer by using the method of undetermined coefficients.

1. $y'' - 5y' + 6y = 2e^t$	2. $y'' - y' - 2y = 2e^{-t}$
3. $y'' + 2y' + y = 3e^{-t}$	4. $4y'' - 4y' + y = 16e^{t/2}$

In each of Exercises 5 through 12 find the general solution of the given differential equation. In Exercises 11 and 12, g is an arbitrary continuous function.

5.	$y'' + y = \tan t,$	$0 < t < \pi/2$
6.	$y'' + 9y = 9\sec^2 3t,$	$0 < t < \pi/6$
7.	$y'' + 4y' + 4y = t^{-2}e^{-2t},$	$t > 0$
8.	$y'' + 4y = 3\csc 2t,$	$0 < t < \pi/2$
9.	$4y'' + y = 2\sec(t/2),$	$-\pi < t < \pi$
10.	$y'' - 2y' + y = e^t/(1 + t^2)$	
11.	$y'' - 5y' + 6y = g(t)$	12. $y'' + 4y = g(t)$

13. Find the general solution of

$$x^2y'' - 6xy' + 10y = 3x^4 + 6x^3,$$

given that $y = x^2$ and $y = x^5$ are linearly independent solutions of the corresponding homogeneous equation.

14. Find the general solution of

$$(x+1)2y'' - 2(x+1)y' + 2y = 1,$$

given that $y = x + 1$ and $y = (x+1)^2$ are linearly independent solutions of the corresponding homogeneous equation.

15. Find the general solution of

$$(x^2 + 2x)y'' - 2(x+1)y' + 2y = (x+2)^2,$$

given that $y = x + 1$ and $y = x^2$ are linearly independent solutions of the corresponding homogeneous equation.

16. Find the general solution of

$$x^2y'' - x(x+2)y' + (x+2)y = x^3,$$

given that $y = x$ and $y = xe^x$ are linearly independent solutions of the corresponding homogeneous equation.

References

1. Robert Alexander Frazer, William Jolly Duncan, Arthur Roderich Collar - Elementary Matrices And Some Applications To Dynamics And Differential Equations - Cambridge University Press-(1947)
2. Haberman R. - Elementary Applied Partial Differential Equations with Fourier Series and Boundary Value Problems-(1987).
3. C. Henry Edwards, David E. Penney - Elementary Differential Equations-Prentice Hall (2007).
4. Henry Thomas Herbert Piaggio - An elementary treatise on differential equations and their applications- G. Bell and Sons, LTD. London (1929).
5. Morris Tenenbaum, Harry Pollard - Ordinary differential equations: an elementary textbook for students of mathematics, engineering, and the sciences-Dover Publications, (1985).
6. Patrick J. Murphy - Elementary Matrices and Some Applications to Dynamics and Differential Equations-Cambridge University Press (1960).
7. Shepley L. Ross - Introduction to Ordinary Differential Equations, 4th Edition-Wiley (1989).
8. William E. Boyce, Richard C. DiPrima - Elementary Differential Equations And Boundary Value Problems-Wiley (2000).