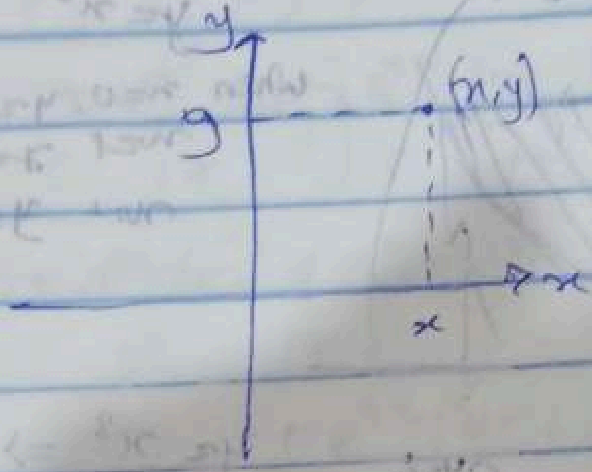
OAU MTH 102 Class Notes

Coordinate Geometry - Probability

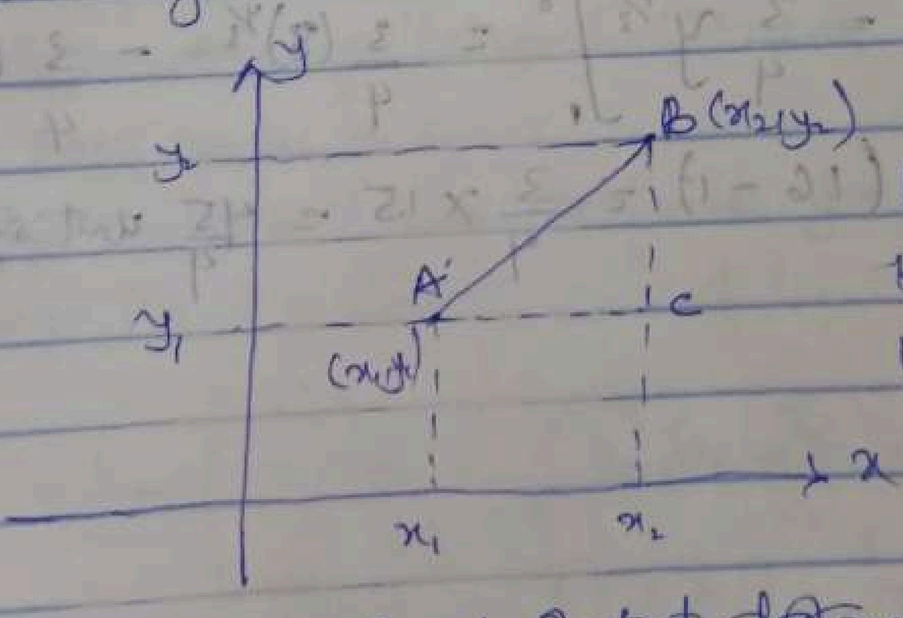
Note Credit: Adebayo Zainab (CHM Department)

Co-ordinate Geometry

- Distance btw 2 points
- Area of triangle
- Division of a straight line



Every point on a plane can be projected horizontally & vertically



$|AB|^2 = |AC|^2 + |BC|^2$

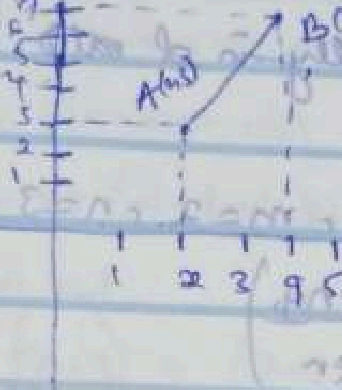
$|AB|^2 = (y_2-y_1)^2 + (x_2-x_1)^2$

$|AB| = \sqrt{(y_2-y_1)^2 + (x_2-x_1)^2}$

A straight line is d shortest distance btw 2 points

$|AB| = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}$

Example :- find the distance btw of points A(2,3) & B(4,7).



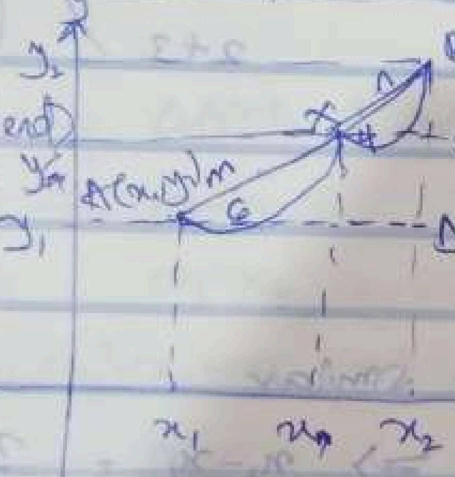
$|AB| = \sqrt{(4-2)^2 + (7-3)^2}$

$= \sqrt{(2)^2 + (4)^2}$

$= \sqrt{4+16} = \sqrt{20} = 2\sqrt{5}$ unit

Division of a straight line

Internal = (distance from the end point to start point)

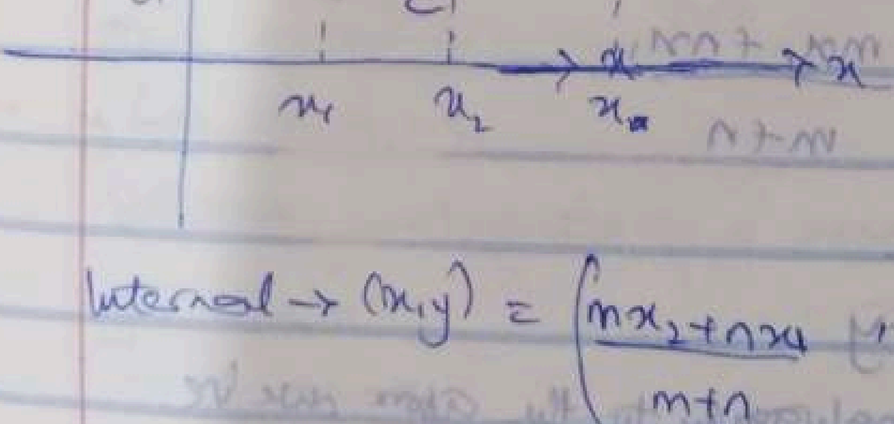


Straight line can be divided internally/externally. A point and same straight line can divide that same straight

m:n

internal 6:4 = 3:2

External -> divides the line externally not btw the 2 points; distance btw the end point & that given point



Internal -> $(x,y) = \left(\dfrac{mx_2+nx_1}{m+n}, \dfrac{my_2+ny_1}{m+n}\right)$

Q. Example: find the points that divides of the joining points $A(2,3)$ and $B(4,7)$ internally in of ratio $(2,3)$.

Sln

$x_1 = 2$, $x_2 = 4$, $y_1 = 3$, $y_2 = 7$, $m = 2$, $n = 3$

$$(x,y) = \left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}\right)$$

$$= \left(\frac{2\times 4 + 3\times 2}{2+3}, \frac{2\times 7 + 3\times 3}{2+3}\right) = \left(\frac{8+6}{5}, \frac{14+9}{5}\right)$$

$$= \left(\frac{14}{5}, \frac{23}{5}\right)$$

$\triangle ABD$, $\triangle XBC$ are similar

$$\frac{m+n}{x_2 - x_1} = \frac{n}{x_2 - x} \Rightarrow \frac{x_2 - x_1}{m+n} = \frac{x_2 - x}{n}$$

$$nx_2 - nx_1 = (m+n)x_2 - (m+n)x$$

$$(m+n)x = (m+n)x_2 - nx_2 + nx_1$$

$$(m+n)x = mx_2 + nx_1$$

$$x = \frac{mx_2 + nx_1}{m+n}$$

$$\frac{m+n}{y_2 - y_1} = \frac{n}{y_2 - y}$$

solving analogously to the above we ve

$$y = \frac{my_2 + ny_1}{m+n}$$

N.B → In the case of midpoint with an internal division

$m = n = 1$

$$\therefore (x, y) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)$$

Considering ΔAXD & ΔABC [which we assume similar]

$$\frac{m}{x - x_1} = \frac{m-n}{x_2 - x_1}$$

$$\frac{x - x_1}{m} = \frac{x_2 - x_1}{m-n} \Rightarrow m(x_2 - x_1) = (x - x_1)(m-n)$$

$$= (m-n)x - (m-n)x_1$$

$$m(x_2 - x_1) = (m-n)x - (m-n)x_1$$

$$\Rightarrow (m-n)x = m(x_2 - x_1) + (m-n)x_1$$

$$x = \frac{mx_2 - mx_1 + mx_1 - nx_1}{m-n} = \frac{mx_2 - nx_1}{m-n}$$

$$\frac{m}{y - y_1} = \frac{m-n}{y_2 - y_1}$$

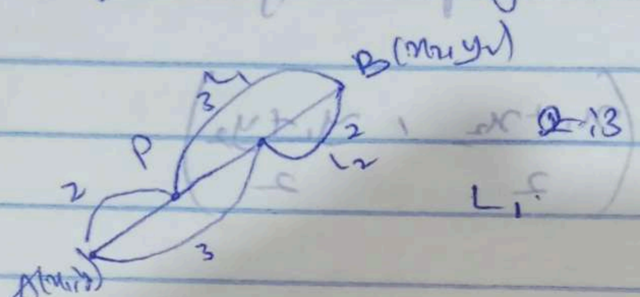
Solving analogously to the above, we have

$$y = \frac{my_2 - ny_1}{m-n}$$

$$(x, y) = \left(\frac{mx_2 - nx_1}{m-n}, \frac{my_2 - ny_1}{m-n}\right)$$ ~~is~~ $\underline{d}$ Co-ordinates

~~Of~~ $\underline{d}$ points that divides line AB externally in $\underline{d}$ ratio m : n

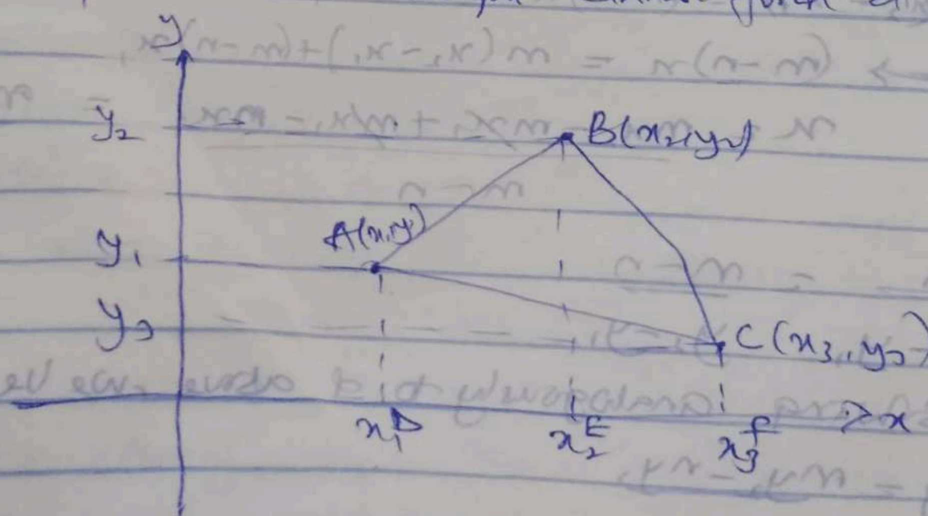
Same angle & same straight line (determines $\underline{d}$ similarity of Δ



Area of triangle

Area of Δ formed by points $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$.

three points on a straight cannot form a Δ



Whenever 3 points are not collinear we've a Δ

Area of $\Delta ABC = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$

$= \frac{1}{2}\left(x_1(y_2-y_3) - x_2(y_1-y_3) + x_3(y_1-y_2)\right)$

$= \frac{1}{2}\left[x_1(y_2-y_3) + x_2(y_3-y_1) + x_3(y_1-y_2)\right]$

$\begin{Bmatrix} 1\ 2\ 3 \\ 2\ 3\ 1 \\ 3\ 1\ 2 \end{Bmatrix}$

2 3 1

Ex:- Find the area of triangle formed by $(-1,0)$, $(3,4)$ & $(2,3)$.

Soln

$\Delta = \frac{1}{2}\cancel{[x_1y_2}- \frac{1}{2}\begin{vmatrix} -1 & 0 & 1 \\ 3 & 4 & 1 \\ 2 & 3 & 1 \end{vmatrix} = \frac{1}{2}\begin{bmatrix} -1(4-3)-0(3-2) \\ +1(9-8) \end{bmatrix}$

$= \frac{1}{2}\left[-1+1\right] = 0$

$\therefore$ Area of d Δ is 0, bcuz d points A, B, C are collinear.

When we've the same gradient (i.e. the gradient of 3 lines), & area is 0 (i.e. d points are collinear).

Take d absolute value of a -ve determinant, bcuz area can not be -ve.

Proof

Area = Area of [ABED + BCFE - ACFD]

Area of $\Delta ABC = \left[\frac{1}{2}(x_2-x_1)(y_1+y_2)\right] + \left[\frac{1}{2}(x_3-x_2)(y_2+y_3)\right]$

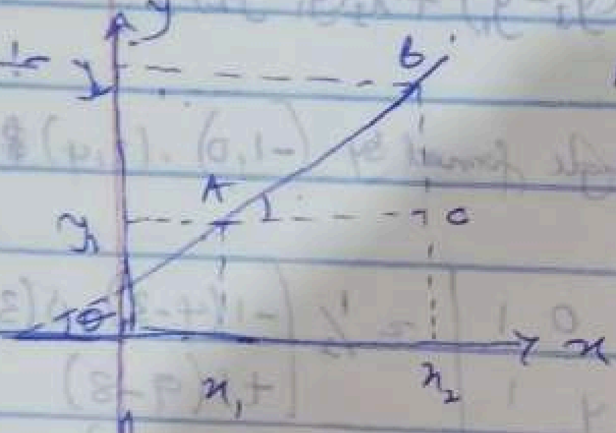
Area of trapezium

$\Delta \ldots - \left(\frac{1}{2}(x_3-x_1)(y_1+y_3)\right)$

$= \frac{1}{2}\left[x_1(y_1+y_3-y_1-y_2) + x_2(y_1+y_2-y_2-y_3) + x_3(y_2+y_3-y_3-y_1)\right]$

$= \frac{1}{2}\left[x_1(y_3-y_2) + x_2(y_1-y_3) + x_3(y_2-y_1)\right]$

Ex :-



$m = \tan\theta$ [using SOH CAH TOA]

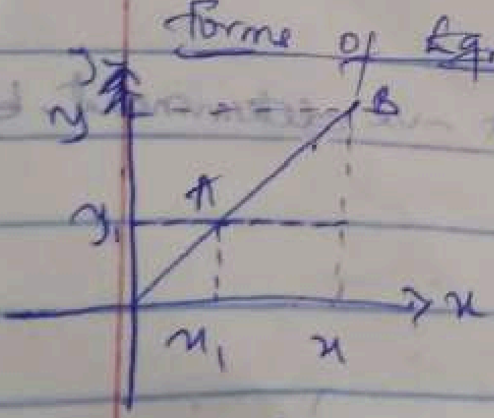
$m = \tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{|BC|}{|AC|}$

$m = \frac{y_2-y_1}{x_2-x_1}$ → gradient form

Ex :- Find the gradient of the line joining point A(2,0) & B(-2,4)

$m = \frac{y_2-y_1}{x_2-x_1} = \frac{4-3}{-2-2} = \frac{1}{-4} = -\frac{1}{4}$

Forms of Eqn of a Straight Line



$m = \tan\theta = \frac{y-y_1}{x_2-x_1}$ (Eqn form)

Ex :- Find the eqn of the line passing through A(-2,3) & B(4,2)

$m = \frac{2-3}{4+2} = -\frac{1}{6}$

$\frac{y-y_1}{x-x_1} = m \Rightarrow \frac{y+3}{x+2} = \frac{-1}{6} \Rightarrow 6y - 18 = -x - 2$

$x + 6y - 16 = 0$

General formula of Eqn of a straight line

$ax + by + c = 0$; where a, b, c are constant.

Ex: find the eqn of a straight line through $A(-4,-1)$ & $B(2,-3)$ and express it in general form.

Soln

$m = \frac{-3+1}{2+4} = \frac{-2}{6} = -\frac{1}{3}$; $\frac{y+1}{x+4} = -\frac{1}{3} \Rightarrow 3y+3 = -x-4$

$x + 3y + 7 = 0$

forms of eqn on a straight line

Recall that;

$\frac{y-y_1}{x-x_1} = m \therefore y - y_1 = m(x - x_1) \Rightarrow y - y_1 = mx - mx_1$

$c = y_1 - mx_1$

$y = mx + y_1 - mx_1 \Rightarrow \boxed{y = mx + c}$ where c is the intercept & m gradient

Gradient intercept form.

Ex: find the eqn of a straight line that passes through point $A(2,3)$ and cut y-axis at 3

Soln

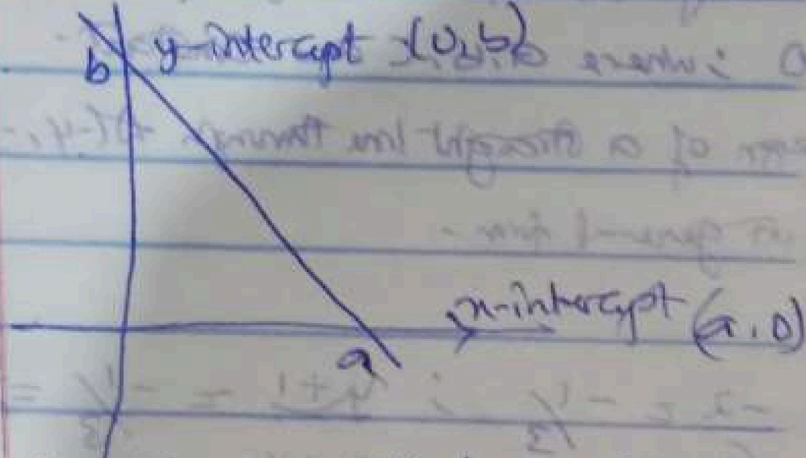
$A(2,3)$, $y = mx + c$, $c = 3$

$y = mx + 3$; $3 = 2m + 3$

$2m = 0$, $m = 0$

$\therefore y = 0x + 3 \Rightarrow y = 3$

Intercept form



$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{0 - b}{a - 0} = \frac{-b}{a}$$

$$\frac{-b}{a} = \frac{y - b}{x - 0} \Rightarrow ay - ab = -bx$$

$$ay + bx = ab \Rightarrow \frac{ay}{ab} + \frac{bx}{ab} = \frac{ab}{ab}$$

$$\frac{y}{b} + \frac{x}{a} = 1$$

$$\frac{x}{a} + \frac{y}{b} = 1$$ - Intercept form

Ex: Find the eqn of the straight line which makes intercept $-3/2$ on x-axis and 7 at the y-axis.

Soln

$a = -3/2$, $b = 7$

$\frac{x}{-3/2} + \frac{y}{7} = 1 \Rightarrow \frac{-2x}{3} + \frac{y}{7} = 1 \Rightarrow -14x + 3y = 21$

$14x - 3y = -21$. $[ax + by + c = d]$

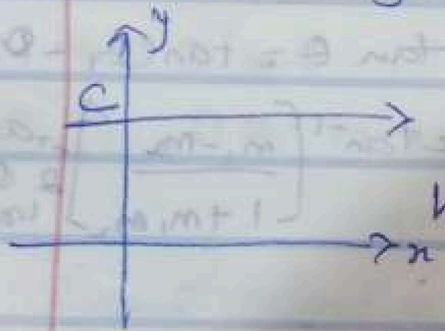
$14x - 3y + 21 = 0$

Using Gradient-Intercept form

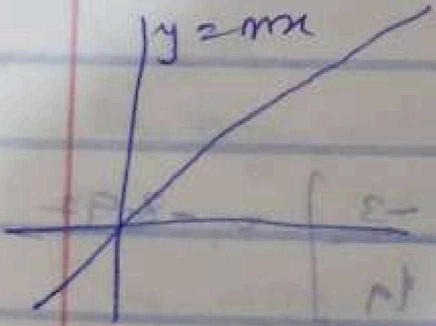
$y = mx + c$

Consider three (3) cases:

Case 1:- $m = 0$, $\therefore y = mx + c \Rightarrow y = (0)x + c = c$



Case 2:- $c = 0$ $\therefore y = mx$

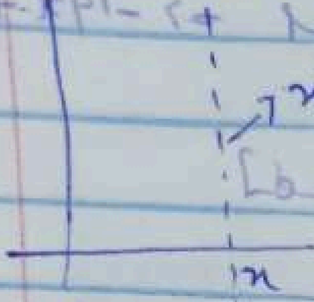


Case 3:- $1/m = 0$

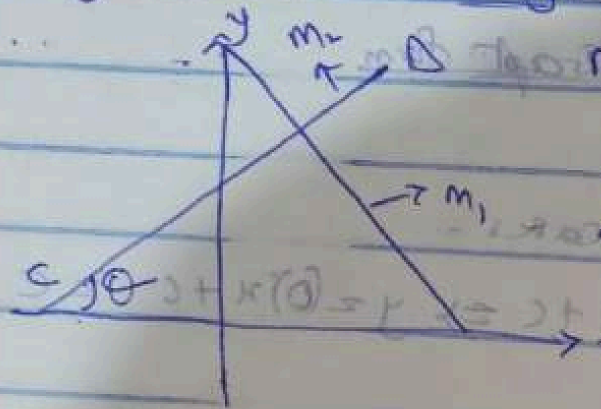
from $\frac{y - y_2}{x - x_1} = m \Rightarrow \frac{1}{m} = \frac{x - x_1}{y - y_1}$; $x = x_1$

$1/m = 0$ $\frac{x - x_1}{y - y_1} = 0 \Rightarrow x - x_1 = 0$, $x = x_1$

Sima :- $y = x = x_1$



Angle btw two straight lines



$m_1 = \tan\theta_1$

$m_2 = \tan\theta_2$

for) $\theta_2 + 180 - \theta_1 = 180$

$\theta = \theta_1 - \theta_2 \rightarrow (1)$

$\tan\theta = \tan(\theta_1 - \theta_2)$

$= \dfrac{\tan\theta_1 - \tan\theta_2}{1 + \tan\theta_1 \tan\theta_2} \Rightarrow \theta = \tan^{-1}\left[\dfrac{m_1 - m_2}{1 + m_1 m_2}\right]$ → angle btw 2 straight line form.

Ex 1 find the angle btw lines $y = 3x + 4$ & $y = 6x - 8$

Soln

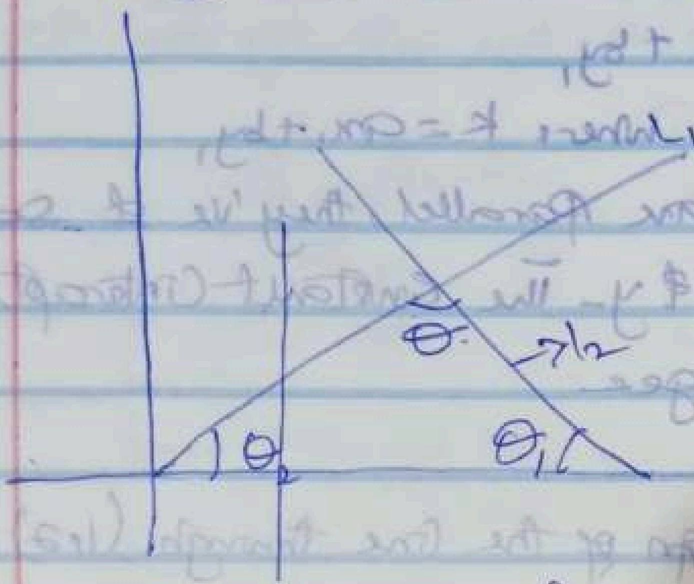
$m_1 = 3$, $m_2 = 6$

$\theta = \tan^{-1}\left[\dfrac{3-6}{1+(3\times 6)}\right] = \tan^{-1}\left[\dfrac{-3}{19}\right] = -8.97$

$\theta = 8.97$ or $180 - 8.97$

$\theta = 171.03^\circ$

Angle btw two Straight lines



$\tan\theta = \dfrac{m_1 - m_2}{1 + m_1 m_2}$

$\tan\theta = \left|\dfrac{m_1 - m_2}{1 + m_1 m_2}\right|$

Special cases

(i) $\theta = 0°$. - two lines that are parallel

$\tan 0° = 0$

$$0 = \frac{m_1 - m_2}{1 + m_1 m_2} \Rightarrow m_1 - m_2 = 0 \Rightarrow m_1 = m_2$$

Example: Find a line that passes through the point (x_1, y_1) and is parallel to the line $ax + by + c = 0$

Soln

The required line $\longrightarrow ax + by + c$

$\longrightarrow$ required

The gradient of the required line will be $m_2 = -a/b$

$by = -ax - c \Rightarrow y = -\frac{a}{b}x - \frac{c}{b}, \ b \neq 0$

The eqn of the required line is the eqn

$$\frac{y - y_1}{x - x_1} = \frac{-a}{b}$$

$by - by_1 = -ax + ax_1$

$ax + by = ax_1 + by_1$

$ax + by = k$, where $k = ax_1 + by_1$

When 2 lines are parallel, they've the same coefficient, i.e x & y. The constant (intercept) is the value that changes

Ex 2:- Find the eqn of the line through $(1, 2)$ which is parallel to the line $4x + 3y + 12 = 0$.

Soln

$4x + 3y + 12 = 0 \Rightarrow 4x + 3y = -12 \Rightarrow 3y = -4x - 12$

$y = \frac{-4}{3}x - \frac{12}{3}$

$m_1 = -4/3$

Since $m_1 = m_2$, m_2, the required gradient $= -4/3$

$\frac{y - y_1}{x - x_1} = m \Rightarrow \frac{y - 2}{x - 1} = \frac{-4}{3}$

$3y - 6 = -4x + 4$

$3y + 4x = 10$

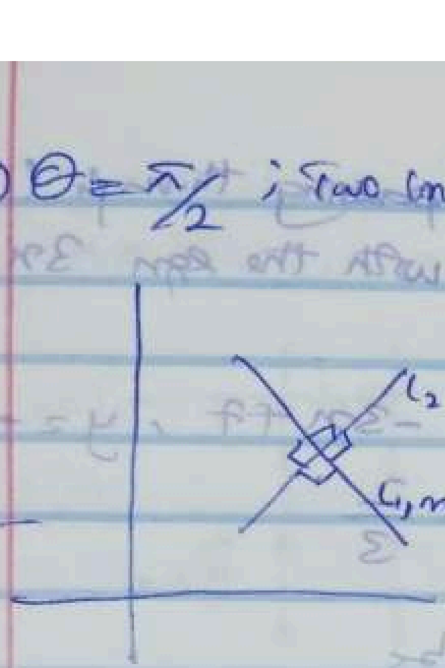
or

$4x + 3y = k$

$k = 4(1) + 3(2) = 4 + 6 = 10$

$\therefore 4x + 3y = 10$

(ii) $\theta = \pi/2$; Two lines that are perpendicular.



$\tan\theta \Rightarrow \tan \pi/2 = \infty$

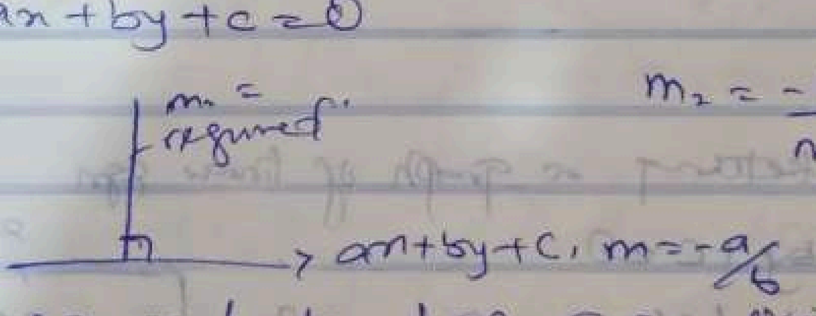
$\infty = \dfrac{m_1 - m_2}{1 + m_1 m_2}$

$1 + m_1 m_2 = 0$

$m_1 m_2 = -1 \Rightarrow m_2 = \dfrac{-1}{m_1}$

Ex 3:- Find a line that passes through the point (x_1, y_1) and is perpendicular to the line -

$ax + by + c = 0$



$m_2 = \dfrac{-1}{m_1} = \dfrac{-1}{\frac{-a}{b}} = \dfrac{b}{a}$

The required line has a gradient of $m_2 = \frac{b}{a}$ and passes through points (x_1, y_1)

i.e. It has eqn

$\dfrac{y - y_1}{x - x_1} = \dfrac{b}{a} \Rightarrow ay - ay_1 = bx - bx_1$

$ay - bx - ay = bx_1 - ay_1$

$bx - ay = k$

where $k = bx_1 - ay_1$

The sign of coefficients of x & y changes

Ex 4 :- find the eqn of the line passing through (1, -2) a perpendicular to the line with the eqn 3x + 2y - 7

Sol

$3x + 2y - 7 = 0$, $2y = -3x + 7$, $y = -\frac{3}{2}x + \frac{7}{2}$

$m_1 = -3/2$, $m_2 = \frac{2}{3}$

The required eqn will be

$2x - 3y = k$

$2(1) - 3(-2) = k \Rightarrow k = 8$

$2x - 3y = 8$

Sketching a graph of linear eqn

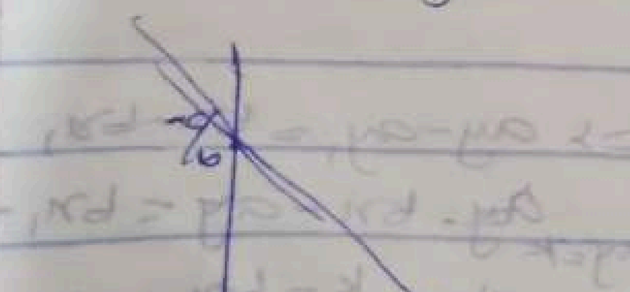
$ax + by + c = 0$

eqn for finding y-intercept

Step 1 :- Let $x = 0$. $y = ?$. $\boxed{y = -c/b}$, $b \neq 0$

Step 2 : Let $y = 0$, $x = ?$. $\boxed{x = -c/a}$, $a \neq 0$

x-intercept



$\boxed{y = mx}$ → eqn dat will pass tru d origin

we've this when $m = +ve$, $y = mx$

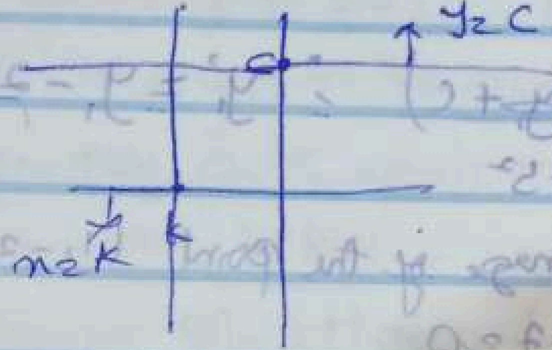
$m = \tan\theta$

when $m = -ve$, $y = mx$

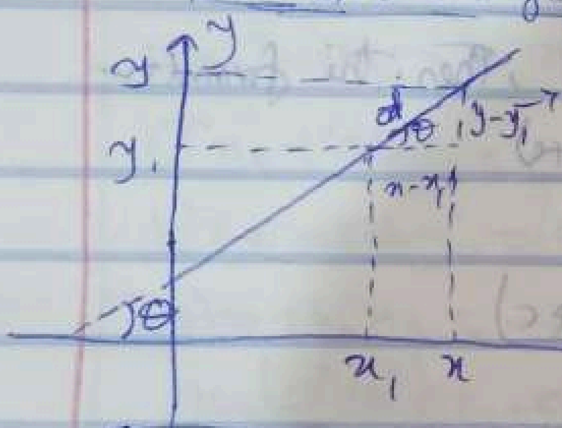
Parametric eqn of

y = c

x = k, c, k are constant



Parametric eqn of a straight line



$\dfrac{y-y_1}{x-x_1} = \tan\theta$

$\dfrac{y-y_1}{x-x_1} = \dfrac{\sin\theta}{\cos\theta}$

$\sin\theta = \dfrac{y-y_1}{d}$, $\cos\theta = \dfrac{x-x_1}{d}$

$y-y_1 = d\sin\theta$

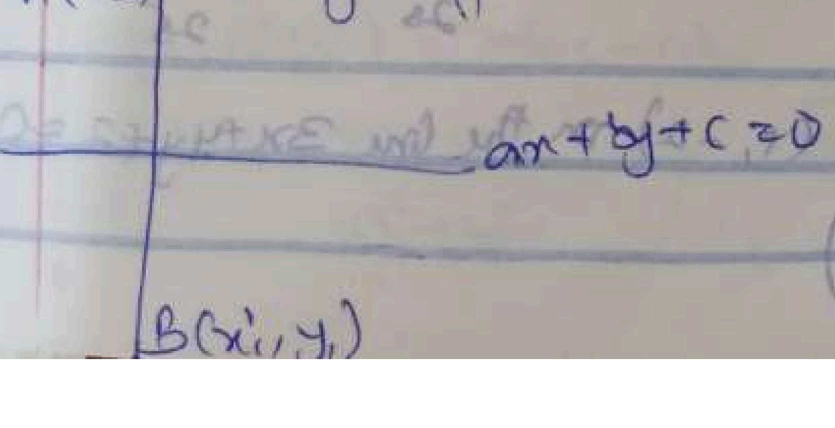
$x-x_1 = d\cos\theta$

$$\boxed{\begin{aligned} y &= y_1 + d\sin\theta \\ x &= x_1 + d\cos\theta \end{aligned}}$$

parametric eqn of straight line

θ - angle maked with the +ve x-axis

Image of a point on a straight line



$$\frac{x_1' - x_1}{a} = \frac{-2(ax_1 + by_1 + c)}{a^2+b^2} = \frac{y_1' - y_1}{b}$$

Note:

$$x_1' = x_1 - \frac{2(ax_1 + by_1 + c)}{a^2+b^2} \; ; \; y_1' = y_1 - \frac{2(ax_1 + by_1 + c)}{a^2+b^2}$$

Ex: Find the mirror image of the point $(1,-2)$ on the line with eqn $3x+4y+7=0$

Soln

Let the image be (α, β), then the formula

$$\alpha = x_1 - \frac{2a(ax_1 + by_1 + c)}{a^2+b^2}$$

$$\beta = y_1 - \frac{2b(ax_1 + by_1 + c)}{a^2+b^2}$$

$a=3, b=4, c=7$

$$\alpha = 1 - \frac{2\times3(3(1)+4(-2)+7)}{3^2+4^2} = 1 - \frac{6(2)}{25} = \frac{25-12}{25}$$

$$\alpha = \frac{13}{25}$$

$$\beta = -2 - \frac{2\times4(2)}{25} = -2 - \frac{16}{25} = \frac{-50-16}{25} = \frac{-66}{25}$$

The image of $(1,-2)$ on the line $3x+4y+7=0$ is

$$\left(\frac{13}{25}, \frac{-66}{25}\right)$$

Proof :- The eqn of the lines are;

$ax + by + c = 0$

$bx - ay - k = 0$

Let (x_0, y_0) be the point of intersection of the lines, i.e.

$ax_0 + by_0 = -c$

$bx_0 - ay_0 = k$

Solving simultaneously, we've

$ax_0 + by_0 = -c \quad \times a$

$bx_0 - ay_0 = k \quad \times b$

$a^2x_0 + aby_0 = -ac$

$b^2x_0 - aby_0 = kb$

$a^2x_0 + b^2x_0 = -ac + kb$

$x_0(a^2+b^2) = -ac + bk \Rightarrow x_0 = \dfrac{bk - ac}{a^2+b^2}$ --- (iii)

$ax_0 + by_0 = -c \quad \times b$

$bx_0 - ay_0 = k \quad \times -a$

$abx_0 + b^2y_0 = -bc$

$-abx_0 + a^2y_0 = -ak$

$a^2y_0 + b^2y_0 = -bc - ak$

$y_0(a^2+b^2) = -bc - ak$

$y_0 = \dfrac{-bc - ak}{a^2+b^2}$ --- (iv)

Note that (x_1, y_1) satisfies the eqn $bx - ay = k$

$bx_1 - ay_1 = k$

$x_0 = \frac{b(bx_1 - ay_1) - ac}{a^2+b^2} = \frac{b^2x_1 - aby_1 - ac}{a^2+b^2}$

Note: (x_0, y_0) is a midpoint of $A(x_1, y_1)$ and $B(x_1', y_1')$

$x_0 = \frac{x_1 + x_1'}{2}$, $y_0 = \frac{y_1 + y_1'}{2}$

Implying

$x_1' = 2x_0 - x_1$

$y_1' = 2y_0 - y_1$

$x_1' = \frac{2(b^2x_1 - aby_1 - ac)}{a^2+b^2} - x = \frac{2b^2x - 2aby - 2ac - a^2x - b^2x}{a^2+b^2}$

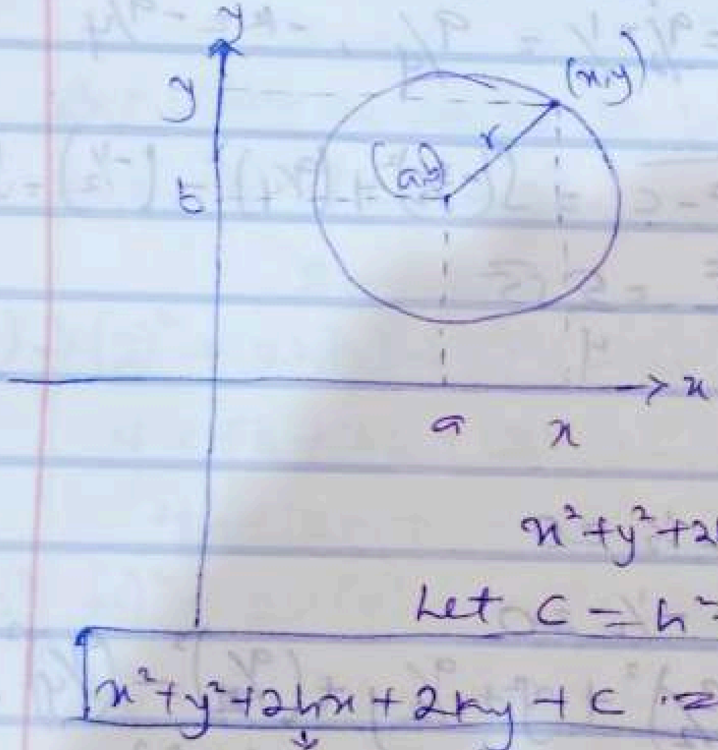
$x_1' = \frac{\cancel{x_1}b^2x_1 + a^2x_1 + 2a^2x_1 - 2aby_1 - 2ac}{a^2+b^2}$

$= \frac{x_1(a^2+b^2) - 2a(ax_1 + by_1 + c)}{a^2+b^2} = \frac{x_1(a^2+b^2)}{a^2+b^2} - \frac{2a(ax_1+by_1}{a^2+b^2}$

$= x_1 - \frac{2a(ax_1 + by_1 + c)}{a^2+b^2}$

Circles

If a point moves subject to a certain restriction, the path is traced out, the locus of a point which moves such that it's always the same distance from a fixed point is called a circle. The fixed point is called the centre of the circle and the constant distance is called the radius of the circle.



$$r = \sqrt{(x-a)^2+(y-b)^2}$$

$$(x-a)^2+(y-b)^2 = r^2$$

$$x^2-2ax+a^2+y^2-2by+b^2=r^2$$

$$x^2+y^2-2ax-2by+a^2+b^2-r^2=0$$

Let $h = -a$, $k = -b$

$$x^2+y^2+2hx+2ky+h^2+k^2-r^2=0$$

Let $c = h^2+k^2-r^2$

$\boxed{x^2+y^2+2hx+2ky+c=0}$ where $c = h^2+k^2-r^2$

Eqn of a circle with centre $(-h,-k)$ and radius $r = \sqrt{h^2+k^2-c}$

* The coefficient of x^2 & y^2 must be the same.
* There's no term in xy
* It is possible there's no term in x & y.
* h^2+k^2-c must be greater than 0 $(a - b)$

Ex:- find the Centre and radius of the Circle whose eqn is given by $2x^2+2y^2-6x+9y-1=0$

Soln

$2x^2+2y^2-6x+9y-1=0$

$x^2+y^2-3x+\frac{9}{2}y-\frac{1}{2}=0$

Centre $=\left(\frac{3}{2}, -\frac{9}{4}\right)$

$2hx=-3x,\ h=-\frac{3}{2},\ -h=\frac{3}{2}$

$2ky=\frac{9}{2}y,\ k=\frac{9}{2}\times\frac{1}{2}=\frac{9}{4},\ -k=-\frac{9}{4}$

Radius, $r=\sqrt{h^2+k^2-c}=\sqrt{\left(-\frac{3}{2}\right)^2+\left(\frac{9}{4}\right)^2-\left(-\frac{1}{2}\right)}=\sqrt{\frac{9}{4}+\frac{81}{16}+\dots}$

$=\sqrt{\frac{125}{16}}=\frac{5\sqrt{5}}{4}=\frac{5\sqrt{5}}{4}$

Or

$x^2+y^2-3x+\frac{9}{2}y-\frac{1}{2}=0$

$x^2-3x+y^2+\frac{9}{2}y-\frac{1}{2}=0$

$x^2-3x+\left(-\frac{3}{2}\right)^2-\left(-\frac{3}{2}\right)^2+y^2+\frac{9}{2}y+\left(\frac{9}{4}\right)^2-\left(\frac{9}{4}\right)^2-\frac{1}{2}=0$

$\left(x-\frac{3}{2}\right)^2+\left(y+\frac{9}{4}\right)^2=\left(-\frac{3}{2}\right)^2+\frac{1}{2}+\left(\frac{9}{4}\right)^2$

$\left(x-\frac{3}{2}\right)^2+\left(y+\frac{9}{4}\right)^2=\frac{125}{16}$

$\left(x-\frac{3}{2}\right)^2+\left(y+\frac{9}{4}\right)^2=\left(\frac{5\sqrt{5}}{4}\right)^2$

$(x-a)^2+(y-b)^2=r^2$

Case 1 :- three given points that the circle passes through

Let $A(x_1, y_1)$, $B(x_2, y_2)$ & $C(x_3, y_3)$

Suppose

$x^2 + 2hx + y^2 + 2ky + c = 0$

$A(x_1, y_1) \to x_1^2 + 2hx_1 + y_1^2 + 2ky_1 + c = 0$ — (1) Solve simultaneously

$B(x_2, y_2) \to x_2^2 + 2hx_2 + y_2^2 + 2ky_2 + c = 0$ — (2)

$C(x_3, y_3) \to x_3^2 + 2hx_3 + y_3^2 + 2ky_3 + c = 0$ — (3)

Ex :- Obtain the eqn of the circle through the 3 points $A(2, 6)$, $B(4, -2)$ & $C(-2, 2)$.

Soln

$A(2, 6) \to (2)^2 + 2h(2) + (6)^2 + 2k(6) + c = 0$

$4 + 4h + 36 + 12k + c = 0$

$4h + 12k + c = -40$ — (i)

$B(4, -2)$ $(4)^2 + 2h(4) + (-2)^2 + 2k(-2) + c = 0$

$16 + 8h + 4 - 4k + c = 0$

$8h - 4k + c = -20$ — (ii)

$C(-2, 2)$ $(-2)^2 + 2h(-2) + (2)^2 + 2k(2) + c = 0$

$4 - 4h + 4 + 4k + c = 0$

$-4h + 4k + c = -8$ — (iii)

$4h + 12k + c = -40$ — ①

$8h - 4k + c = -20$ — ②

$-4h + 4k + c = -8$ — ③

Add eq ① & ②

$16k + 2c = -48$ — ④

Multiply eq 3 by 2 & add with eq ②

$4k + 3c = -36$ — ⑤ $\times 4$

$16k + 2c = -48$ — ④ $\times -6$

$16k + 12c = -144$ — ⑥

$-10c = 96$

$c = \frac{96}{-10} = -\frac{96}{10} = -\frac{48}{5}$

Multiply eq 4 by 6

$-80k = +144$

$k = -9/5$

from eqn ③

$-4h + 4k + c = -8$

$-4h + 4\left(-\frac{9}{5}\right) + -\frac{48}{5} = -8$

$-4h = \frac{44}{5}$

$h = -11/5$

$\therefore k = -9/5,\ h = -11/5,\ c = -48/5$

Centre = $(-h, -k) = \left(\frac{11}{5}, \frac{9}{5}\right)$

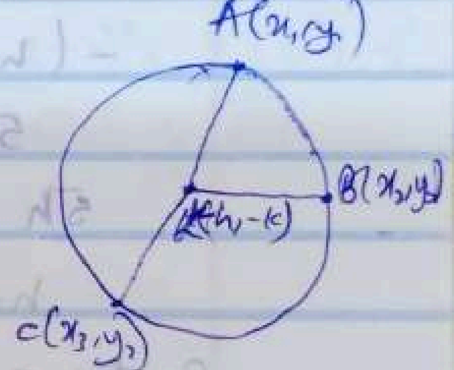
radius = $\sqrt{h^2+k^2-c} = \sqrt{\left(\frac{-11}{5}\right)^2 + \left(\frac{-9}{5}\right)^2 + \frac{48}{5}} = \sqrt{\frac{121}{25}+\frac{81}{25}+\frac{48}{5}}$

$= \sqrt{\frac{202}{25}+\frac{48}{5}} = \sqrt{\frac{202+240}{25}} = \sqrt{\frac{442}{25}} = \frac{\sqrt{442}}{5}$

$x^2+2hx+y^2+2ky+c=0$

$x^2 - \frac{22}{5}x + y^2 - \frac{18}{5}y - \frac{48}{5} = 0$

$5x^2 - 22x + 5y^2 - 18y - 48 = 0$



Method 2:-

$|KA| = |KB|$ — ①

$|KA| = |KC|$ — ②

Ex: $A(2,6)$, $B(4,-2)$; $C(-2,2)$

Solution: Let $K = (-h,-k)$

$|AK| = \sqrt{(2+h)^2+(6+k)^2}$

$|KB| = \sqrt{(4+h)^2+(-2+k)^2}$

$|KA| = |KB| \rightarrow (2+h)^2+(6+k)^2 = (4+h)^2+(-2+k)^2$

$\Rightarrow \cancel{4}+4h+\cancel{h^2}+36+12k+\cancel{k^2} = 16+8h+\cancel{h^2}+\cancel{4}-4k+k^2$

$4h+12k+4k-8h-16+36$ ①

$-4h+16k = -20$

$h-4k = 5$

$|KC| = \sqrt{(-2+h)^2+(k+2)^2}$

$|KC| = |KA| \Rightarrow \cancel{4}+\cancel{h^2}+\cancel{4h}^{-4h}+\cancel{k^2}+4k+4 = \cancel{4}+4h+\cancel{k^2}+36+12k+\cancel{k^2}$

~~$4h+4k+4 \quad 4k+4-36-12k=0$~~

$-8k-32=0 \quad -4h+4k+4-4h-36-12k=0$

$8k=-32, \; k=-32/8=$

$-8h-8k-32=0$

$-h-k-4=0$

$h+k=-4 \quad — (2) \quad \times 4$

$-(h-4k=5 \quad — (1)$

$5k=-9, \; k=-9/5$

$5h=-11$

$h=-11/5$

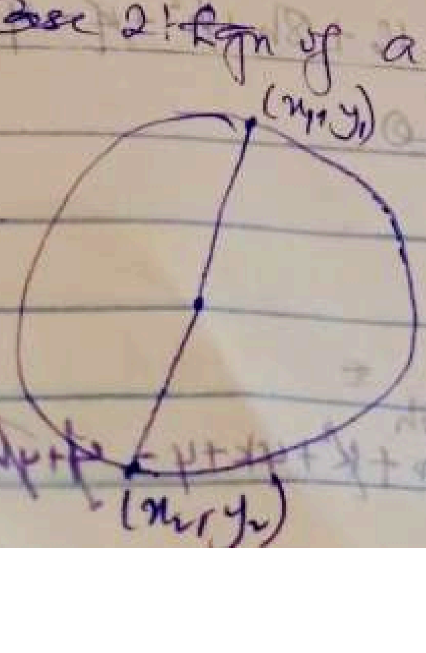
Centre $=(-h,-k) = (11/5, 9/5)$

r = distance btw $(11/5, 9/5)$ and any of the given point $(2,6)$

$r=\sqrt{(2-11/5)^2+(6-9/5)^2} = \sqrt{\left(\frac{-1}{5}\right)^2+\left(\frac{21}{5}\right)^2} = \sqrt{\frac{1}{25}+\frac{441}{25}}$

$=\sqrt{\frac{442}{25}} = \frac{\sqrt{442}}{5}$

Case 2: Eqn of a circle given a diameter



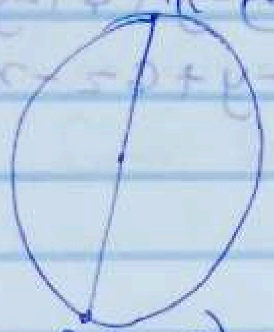
$x^2+y^2+2hx+2ky+c=0$

The Centre = midpoint $=\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$

$r=\frac{1}{2}$ diameter $=\frac{1}{2}\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$

Find the eqn of the circle with P_1, P_2 as diameter where $P_1 = (-1, 2)$ and $P_2 = (2, -3)$.

$P_1(-1, 2)$



$P_2(2, -3)$

$$\text{Centre} = \left(\frac{-1+2}{2}, \frac{2-3}{2}\right)$$

$$(a, b) = \left(\tfrac{1}{2}, -\tfrac{1}{2}\right)$$

$$r = \frac{1}{2}\sqrt{(2+1)^2 + (-3-2)^2} = \tfrac{1}{2}\sqrt{3^2 + -5^2}$$

$$= \tfrac{1}{2}\sqrt{9+25} = \tfrac{1}{2}\sqrt{34}$$

$$= \frac{\sqrt{34}}{2}$$

$$(x-a)^2 + (y-b)^2 = r^2$$

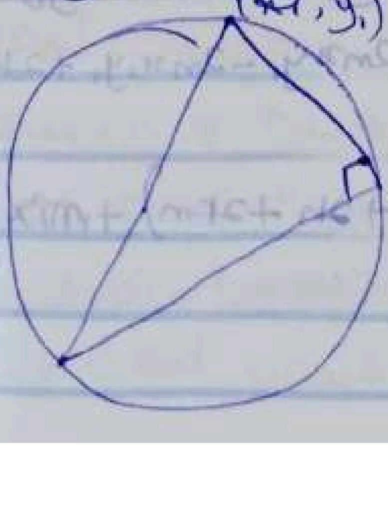
$$\left(x-\tfrac{1}{2}\right)^2 + \left(y+\tfrac{1}{2}\right)^2 = \left(\frac{\sqrt{34}}{2}\right)^2$$

$(a, b) = (-h, -k)$, where, $a = -h$, $b = -k$

$h = -\tfrac{1}{2}$, $k = \tfrac{1}{2}$

$$c = -h^2 \mp k^2 + r^2 = -\tfrac{1}{4} \mp \tfrac{1}{4} + \frac{34}{4} = \frac{32}{4} = 8$$

$$x^2 + y^2 - x + y + 8 = 0$$



$$\left(\frac{y-y_1}{x-x_1}\right) \cdot \left(\frac{y_2-y_1}{x_2-x_1}\right) = -1$$

$P_1 = (-1, 2)$

$P_2 = (2, -3)$

$$\left(\frac{y-2}{x+1}\right)\left(\frac{-3-y}{2-x}\right)$$

$$\frac{(y-2)(-3-y)}{(x+1)(2-x)} = -1 \Rightarrow -3y - y^2 + 6 + 2y = -[2x - x^2 + 2 - x]$$

$$\Rightarrow -y^2 - y + 6 = -x + x^2 - 2$$

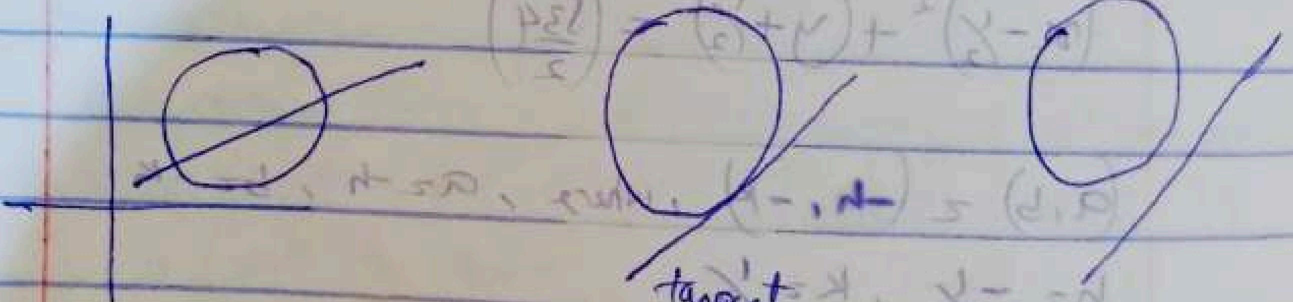
$$-y^2 - y + x - x^2 + 2 + 6 = 0$$

$$x^2 + y^2 - x + y - 8 = 0$$

Intersection of a Circle and a Line

$$x^2 + y^2 + 2hx + 2ky + c = 0 \quad -①$$

$$y - y_1 = m(x - x_1)$$



tangent

$$y = mx - mx_1 + y_1$$

Put y in eq ①

$$x^2 + (mx - mx_1 + y_1)^2 + 2hx + 2k(mx - mx_1 + y_1) + c = 0$$

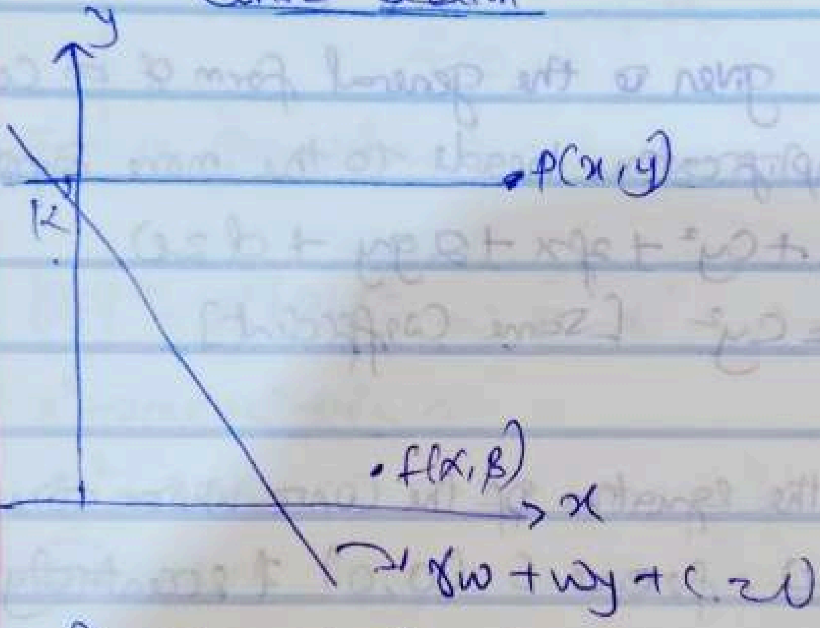
$$x^2 + m^2x^2 + m^2x_1^2 + y_1^2 - 2m^2xx_1 + 2mxy_1 - 2mx_1y_1 + 2hx + 2kmx - 2kmx_1$$
$$+ 2ky_1 + c = 0$$

$$(1+m^2)x^2 + x(2my_1 - 2m^2x_1 + 2h + 2km) + m^2x_1^2 + y_1^2 - 2mx_1y_1$$
$$- 2kmx_1 + 2ky_1 + c = 0$$

discriminant determines the nature of the root of quadratic eqn

$\Delta = b^2 - 4ac = \begin{cases} >0 \\ =0 \\ <0 \end{cases}$

Conic Section



A Conic section is a Curve traced by a point P such that the ratio of the distance btw P and a fixed point F to the distance of point P is equal to a constant say e

Mathematically, a Conic section is defined via

$$\frac{|PF|}{|PK|} = e = \text{Constant}$$

Where PK is the perpendicular distance of P to L

fixed point f (α, β) is called focus.

fixed line L described by $\gamma x + wy + c = 0$ is called directrix

e is called eccentricity.

Recall, $\frac{|PF|}{|PK|} = e$

$\Rightarrow |PF|^2 = e^2|PK|^2$

$(x-\alpha)^2 + (y-\beta)^2 = e^2\left(\frac{\gamma x+\omega y+c}{\gamma^2+\omega^2}\right)^2$

Equation A given is the general form of a Conic Section. further simplifications leads to the more simplified form,

$ax^2 + bxy + cy^2 + 2fx + 2gy + d = 0$

N.B: $ax^2 = cy^2$ [Same Coefficient]

Ex: Obtain the equation of the conic whose directrix is $2x-y+1=0$, focus is $(0,0)$ & eccentricity is $\frac{1}{2}$

Soln

Recall: $|PF|^2 = e^2|PK|^2$, f is $(0,0)$,

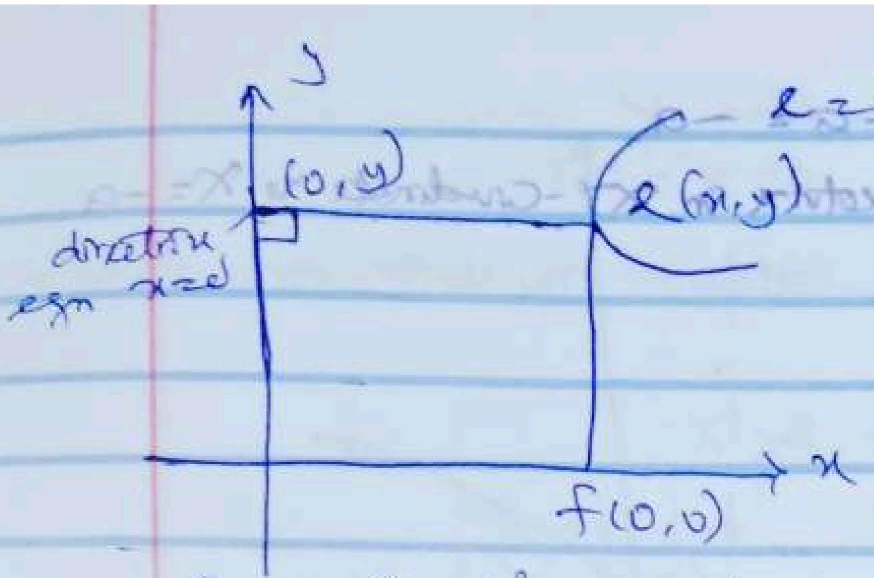
$|PK|^2 = \frac{(2x-y+1)^2}{(2^2+(-1)^2)} \Rightarrow x^2+y^2 = e^2\frac{(2x-y+1)^2}{5} = \left(\frac{1}{2}\right)^2\frac{(2x-y+1)^2}{5}$

$20(x^2+y^2) = (2x-y+1)^2$

$20x^2+20y^2 = 4x^2+y^2+1-4xy-2y+4x$

$16x^2+19y^2+4xy+2y-4x-1=0$

N.B: If $e = 1$, the conic section is parabola



For a Parabola, $e=1$

$|PF|^2 = e^2|PK|^2 \Rightarrow (x-c)^2 + y^2 = x^2$

$x^2 - 2xc + c^2 + y^2 = x^2$

$y^2 = 2xc - c^2 = 2c\left(x - \frac{c}{2}\right)$

The next thing is to transform (x,y) coordinate to (X,Y) coordinate system using:

$X = x - a$ and $Y = y$, where $a = \frac{c}{2}$

$\Rightarrow y^2 = 2c\left(x - \frac{c}{2}\right) = 4ax$

$\boxed{Y^2 = 4aX}$ → the eqn of Parabola in standard form.

Recall that: the focus, directrix, in the xy-coordinate are $(c,0)$ and $x=0$ respectively.

In the new coordinate, We've,

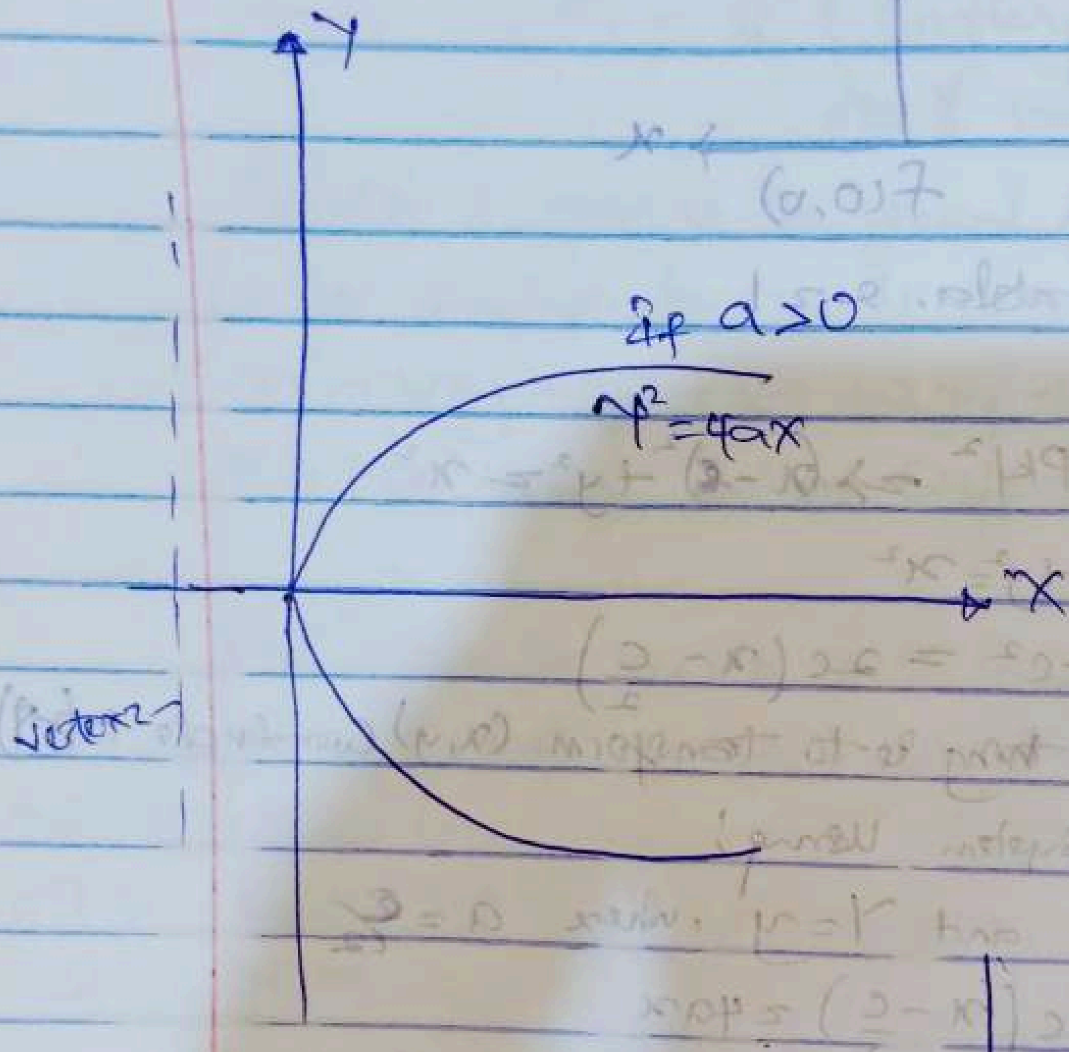
(i) focus: $X = x - a \Rightarrow X = c - a = 2a - a = a$

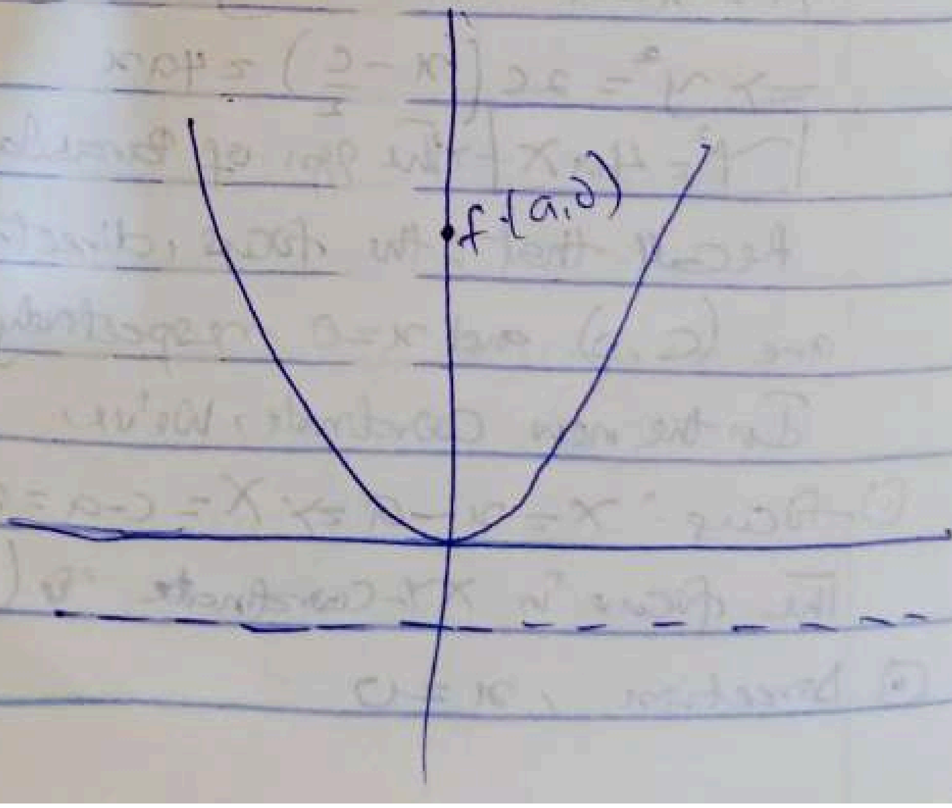
The focus in XY-coordinate is $(a,0)$

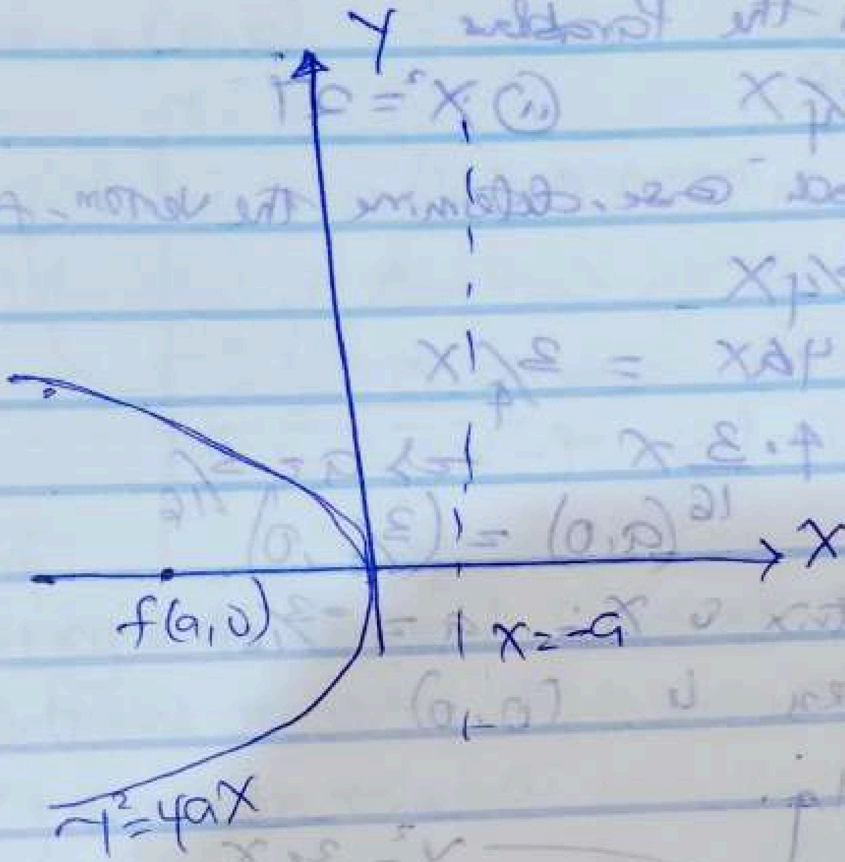
(ii) Directrix, $x=0$

$X = x - a = 0 - a = -a$

Thus, the directrix in XY-coordinate is $X = -a$






Y
X
f(a,0)
x=-a
y²=4ax

Ex1- Sketch the Parabolas

(i) $y^2 = \frac{3}{4}x$ (ii) $x^2 = 2y$

In each case, determine the vertex, focus & directrix.

(i) $y^2 = \frac{3}{4}x$

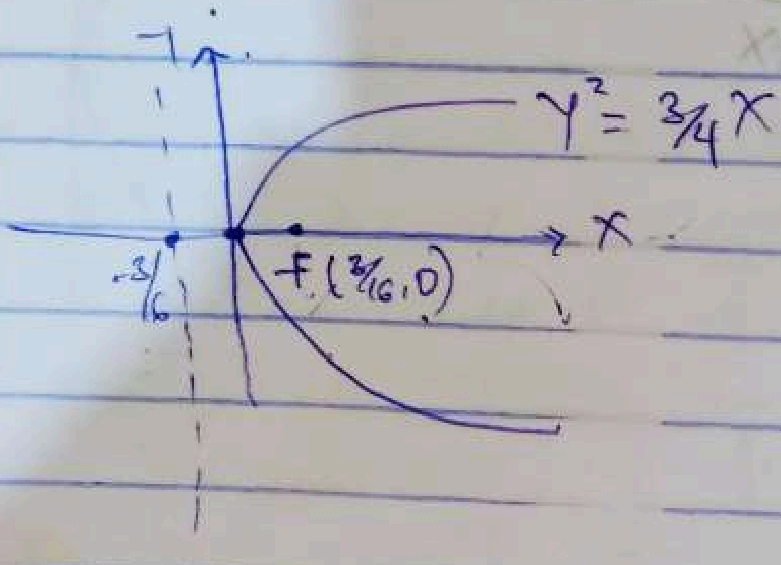
$y^2 = 4ax = \frac{3}{4}x$

$= 4 \cdot \frac{3}{16}x \Rightarrow a = \frac{3}{16}$

focus : $(a, 0) = (\frac{3}{16}, 0)$

directrix : $x = -a = -\frac{3}{16}$

vertex is $(0, 0)$



(ii) $x^2 = 2y = 4ay$

i.e $x^2 = 4ay$

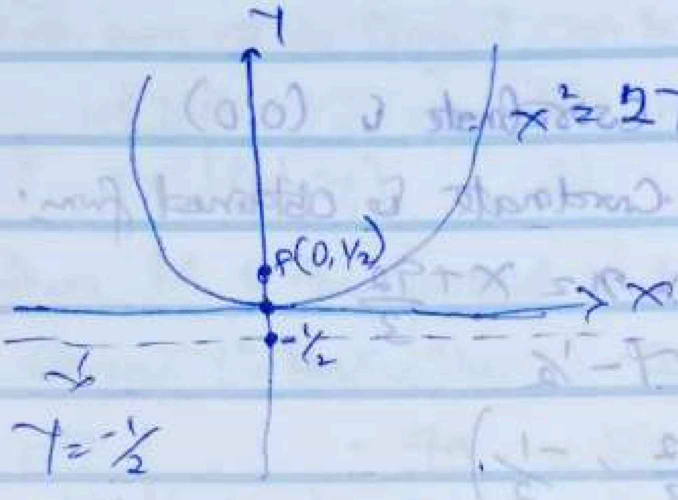
$\Rightarrow x^2 = 4 \cdot \frac{2}{4}y = 4 \cdot \frac{1}{2}y$

$a = \frac{1}{2}$

focus : $(0, a) = (0, \frac{1}{2})$

directrix : $y = -a = -\frac{1}{2}$

vertex $= (0,0)$



<u>Working in the Original Co-ordinates for Parabola</u>

Ex:- Determine the vertex, focus, directrix of the conic

$40y^2 + 40y - 3x + 102 = 0$

<u>Soln</u>

By method of completing the square

$$40(y^2+y) - 3x + 102 = 0$$

$$40\left(\left(y+\tfrac{1}{2}\right)^2 - \tfrac{1}{4}\right) - 3x + 102 = 0$$

$$40\left(y+\tfrac{1}{2}\right)^2 - 3x + 92 = 0$$

$$\left(y+\tfrac{1}{2}\right)^2 = \frac{1}{40}(3x - 92)$$

$$\left(y+\tfrac{1}{2}\right)^2 = \frac{3}{40}\left(x - \frac{92}{3}\right)$$

$$\left(y+\tfrac{1}{2}\right)^2 = \frac{4 \cdot 3}{160}\left(x - \frac{92}{3}\right)$$

Let $Y = y + 1/2$, $X = x - \frac{92}{3}$

$\Rightarrow Y^2 = 4 \cdot \frac{3}{160} X$

Comparing with $Y^2 = 4aX$ we've

$a = \frac{3}{160}$.

Vertex in XY coordinate is $(0,0)$

Vertex in xy coordinate is obtained from:

$X = x - \frac{92}{3} \Rightarrow x = X + \frac{92}{3}$

Similarly $y = Y - \frac{1}{2}$

Vertex is $\left(\frac{92}{3}, -\frac{1}{2}\right)$

Focus in XY coordinate is $(a, 0) = \left(\frac{3}{160}, 0\right)$

In xy coordinate is $\left(X + \frac{92}{3}, Y - \frac{1}{2}\right) = \left(\frac{3}{160} + \frac{92}{3}, 0 - \frac{1}{2}\right)$

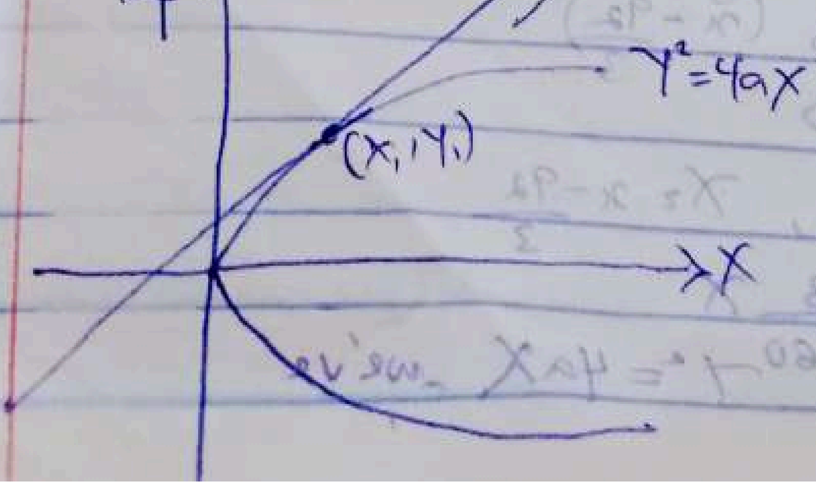
$= \left(\frac{14729}{480}, -\frac{1}{2}\right)$

Directrix in XY is $X = -\frac{3}{160}$

In xy-coordinate, directrix is $x = \frac{-3}{160} + \frac{92}{3}$

$x = \frac{14711}{480}$

Tangent & Normal to the Parabola



Given the parabola $Y^2 = 4aX$ and a tangent at (X_1, Y_1) then eqn of the tangent can be obtained from the fact

$$\left.\frac{dY}{dX}\right|_{(X_1, Y_1)} = M_1$$

Where M_1 is the gradient of tangent at (X_1, Y_1)

Given $Y^2 = 4aX$, then, $2Y\frac{dY}{dX} = 4a$

$$\left.\frac{dY}{dX}\right|_{(X_1, Y_1)} = \frac{4a}{2Y_1} = \frac{2a}{Y_1}$$

Eqn of tangent at (X_1, Y_1)

$Y - Y_1 = M_1(X - X_1)$

$Y - Y_1 = \frac{2a}{Y_1}(X - X_1)$

$YY_1 - Y_1^2 = 2aX - 2aX_1$, since $Y_1^2 = 4aX_1$

$\Rightarrow YY_1 - 4aX_1 = 2aX - 2aX_1$

$YY_1 = 2aX - 2aX_1 + 2aX_1 \Rightarrow YY_1 = 2a(X + X_1)$

$\boxed{YY_1 = 2a(X + X_1)}$ ← Eqn of the tangent

Ex: Obtain the eqn of tangent to the parabola

$y^2 = \frac{3}{4}x$ at $(12, 3)$ $\rightarrow$ $Y^2 = 4ax = 4 \cdot \frac{3}{16}x$

Soln

$(X_1, Y_1) = (12, 3)$ $\quad a = \frac{3}{16}$

Eqn of tangent $\Rightarrow YY_1 = 2a(X + X_1)$

(Y_1, X_1) to $3Y = 2 \times \frac{3}{16}(X + 12)$

$3Y = \frac{6^3}{16_8}X + \frac{24^{12}}{16_8} \Rightarrow 3Y = \frac{3X}{8} + \frac{12}{8}$

$8Y = X + 12$

~~Equation~~ of normal can be obtain from the fact that

$M_2 = \frac{-1}{M_1} = \frac{-Y_1}{2a}$

Where M_2 is the gradient of normal at (X_1, Y_1)

Eqn of normal is

$Y - Y_1 = \frac{-Y_1}{2a}(X - X_1)$

$2aY - 2aY_1 = -Y_1X + Y_1X_1$

$\boxed{2aY + Y_1X = 2aY_1 + Y_1X_1}$ ← Eqn of normal

Exercise: Obtain the eqn of the normal to the parabola $y^2 = \frac{3}{4}x$ at $(12, 3)$

$a = 3/16$, $X_1 = 12$, $Y_1 = 3$

$2 \cdot \frac{3}{16_8}Y + 3X = 2 \cdot \frac{3}{16_8} \cdot 3 + 3 \cdot 12$

$\frac{3Y}{8} + 3X = \frac{9}{8} + 36$

$24\cancel{7} + 3Y + 24X = 9 + 288$

$3Y + 24X = 297$

$Y + 8X = 99$

Parametric Eqn of a Parabola

The Parametric eqn for a parabola $Y^2 = 4aX$ are given by $Y = 2at$ and $X = at^2$; where t is a parameter

Clearly;

$Y^2 = (2at)^2 = 4a^2t^2 = 4a \cdot x \, at^2$

$\therefore Y^2 = 4aX$

Hence; $Y = 2at$ and $X = at^2$ are parametric eqn of the parabola eqn $Y^2 = 4aX$ / Cartesian eqn of a parabola

The eqn $Y^2 = 4aX$ is called the Cartesian eqn of a Parabola.

Ex: Obtain the parametric eqn of the following Conic sections: (i) $(y+2)^2 = 4x - 12$ (ii) $3x^2 = 1 - 2y$

Soln

(i) $(y+2)^2 = 4x - 12 \Rightarrow$

$= 4(x-3)$

Let $Y = y+2$ and $X = x-3$

$Y^2 = 4aX = 4X \Rightarrow a = 1$

Recall: $Y = 2at$ & $X = at^2$

$\Rightarrow Y = 2t$ & $X = t^2$

$\Rightarrow y+2 = 2t$ & $x-3 = t^2$

$\therefore y = 2t-2$ & $x = t^2+3$

(ii) $3x^2 = 1-2y$

Write in the form $X^2 = 4aY$, so that we've parametric eqn as $X = 2at$ & $Y = at^2$

$3x^2 = -(2y-1)$

$3x^2 = -2(y-\frac{1}{2})$

$x^2 = \frac{-2}{3}(y-\frac{1}{2})$

Let $X = x$ & $Y = y - \frac{1}{2}$

$X^2 = -\frac{2}{3}Y$

$X^2 = 4\left(\frac{-2}{12}\right)Y$

$4aY$

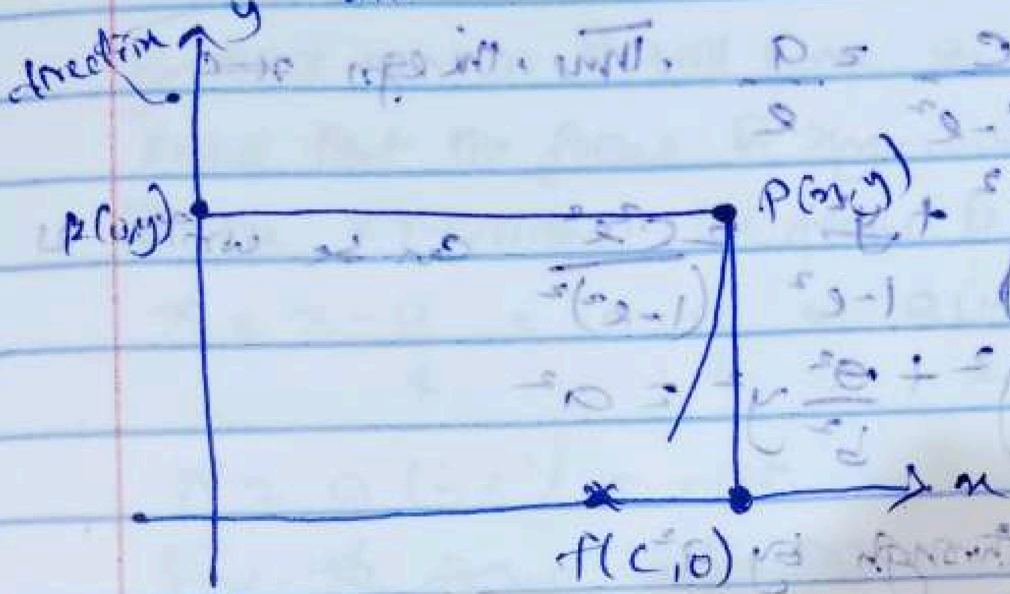
$a = \frac{-2}{12} = \frac{-1}{6}$

Recall that, $X = 2at$ & $Y = at^2$

$X = \frac{-1}{3}t$ & $Y = \frac{-t^2}{6}$

$x = -\frac{1}{3}t$ & $y = \frac{-t^2}{6} + \frac{1}{2} = \frac{1}{2} - \frac{t^2}{6}$

THE ECLIPSE



Here, for an ecllipse, the eccentricity e lies between 0 and 1. Precisely $0 < e < 1$.

From the figure;

$$|PF|^2 = e^2 |PH|^2$$

$$(x-c)^2 + y^2 = e^2 (x^2)$$

$$(1-e^2)x^2 - 2cx + y^2 = -c^2$$

$$\Rightarrow x^2 - \frac{2c}{1-e^2}x + \frac{y^2}{1-e^2} = \frac{-c^2}{1-e^2}$$

$$\left(x - \frac{c}{1-e^2}\right)^2 + \frac{y^2}{1-e^2} = \frac{c^2}{(1-e^2)^2} - \frac{c^2}{1-e^2}$$

$$\Rightarrow \left(x - \frac{c}{1-e^2}\right)^2 + \frac{y^2}{1-e^2} = \frac{c^2 e^2}{(1-e^2)^2}$$

Let $a^2 = \dfrac{c^2e^2}{(1-e^2)^2}$ and $b^2 = a^2(1-e^2)$, so that,

$$\frac{1}{1-e^2} = \frac{a^2}{b^2}$$

If $a, b > 0$,

then $\frac{c}{1-e^2} = \frac{a}{e}$. Thus, the eqn

$$\left(x-\frac{c}{1-e^2}\right)^2 + \frac{y^2}{1-e^2} = \frac{c^2e^2}{(1-e^2)^2}$$ can be written as

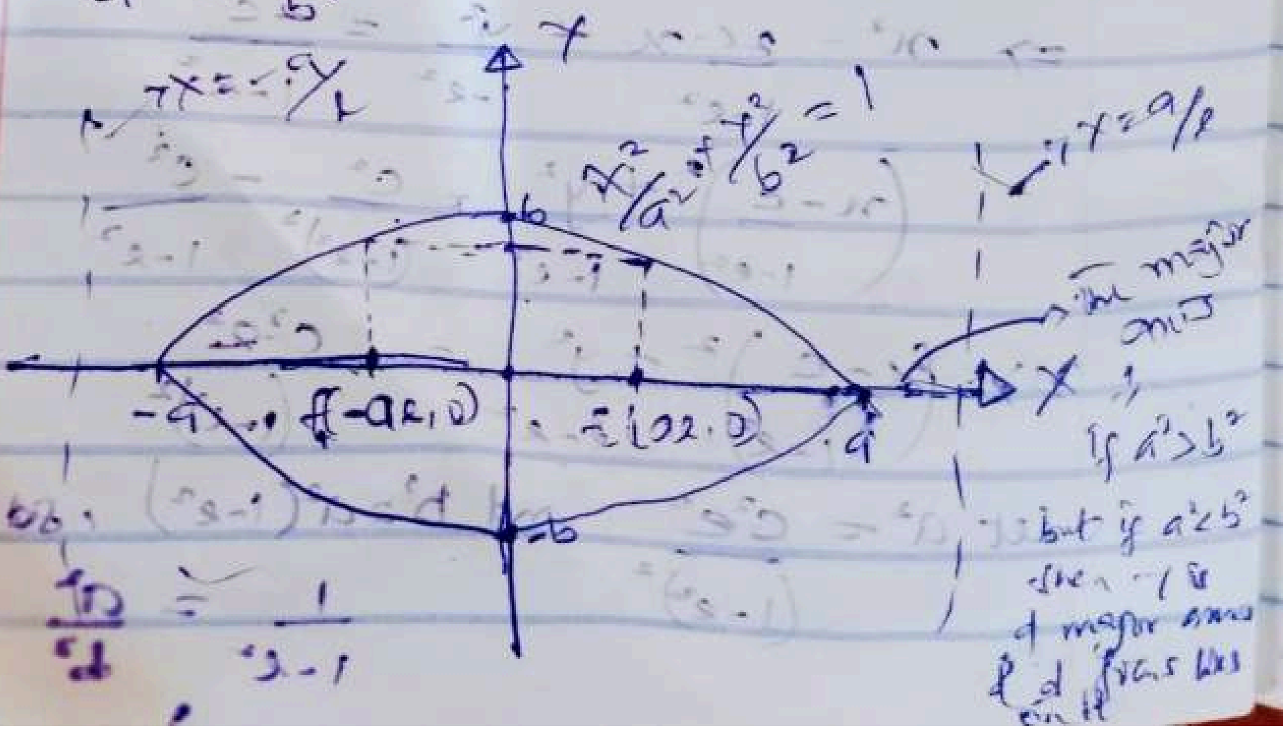
$$\left(x-\frac{a}{e}\right)^2 + \frac{a^2}{b^2}y^2 = a^2$$

Divide through by a^2.

$$\frac{\left(x-\frac{a}{e}\right)^2}{a^2} + \frac{y^2}{b^2} = 1$$

Let $X = x - \frac{a}{e}$ and $Y = y$, then the eqn of an ellipse in standard form is written as:

$$\frac{X^2}{a^2} + \frac{Y^2}{b^2} = 1$$



$b^2 < a^2$, $Y = \pm b$, the ellipse is symmetrical, for this reason, it will have 2 foci.
Recall that the focus in xy-coordinate is $(c, 0)$
In the XY-coordinate, the focus is obtained from the

$$X = x - \frac{a}{e} = c - \frac{a}{e} = \frac{a(1-e^2)}{e} - \frac{a}{e} = \frac{a}{e}(1-e^2-1)$$

$$X = \frac{a}{e}(-e^2) = -ae$$

Thus, the focus in the XY-coordinate is $(-ae, 0)$
By the fact that the ellipse is symmetry above Y-axis, the other focus is $(+ae, 0)$
The reason of focus is inside and not outside is bcoz we multiplied e which is < 1 by a which contract to give a smaller number to be inside.

Directrix in the xy-coordinate is $x = 0$ and in the XY-coordinate, we've

$X = x - \frac{a}{e}$, $X = -\frac{a}{e} \longrightarrow$ Directrix
The other directrix (bcoz of ellipse is symmetric) is

$$X = \frac{a}{e}$$

Lecture 04/05/23

Example: Obtain the centre, vertices, ~~focus~~ foci and directrices of the following

(i) $16x^2 + 25y^2 = 400$

(ii) $5x^2 + 4y^2 = 20$

Solution

(i) $16x^2 + 25y^2 = 400$

divide through by 400

$$\frac{x^2}{25} + \frac{y^2}{16} = 1$$

By comparing with standard form;

$a^2 = 25$ and $b^2 = 16$

Recall that

$$b^2 = a^2(1 - e^2)$$

$$16 = 25(1 - e^2)$$

$$e = 3/5$$

Whenever the ellipse is written in standard form, the center is (0, 0)

Vertices are $(-a, 0)$ and $(a, 0)$

$a^2 = 25, \quad a = 5$

So that the vertices are $(-5, 0)$ and $(5, 0)$

Foci are $(-ae, 0)$ and $(ae, 0)$

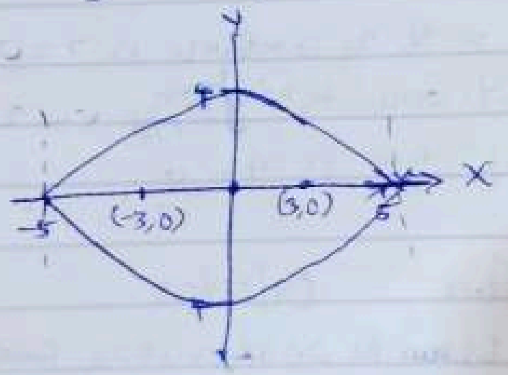
$$ae = 5 \times \frac{3}{5} = 3$$

foci are $(-3, 0)$ $(3, 0)$

Directrices are $x = -\frac{a}{e}$ and $\frac{a}{e}$

$$\frac{a}{e} = \frac{5 \times 5}{3} = \frac{25}{3}$$

$$x = -\frac{25}{3} \text{ and } \frac{25}{3}$$



① $5x^2 + 4y^2 = 20$

Divide through by 20

$$\frac{x^2}{4} + \frac{y^2}{5} = 1$$ (Here $a^2 \neq 4$ because a^2 must be $> b^2$. $\therefore a^2 = 5$, the graph will focus on the y axis.)

$a^2 = 5$ and $b^2 = 4$.

This, compared to $\frac{y^2}{a^2} + \frac{x^2}{b^2} = 1$

$a = \sqrt{5}, \quad b = 2$

from; $b^2 = a^2(1 - e^2)$

$$4 = 5(1 - e^2)$$

$$e = \frac{1}{\sqrt{5}}$$

Centre is $(0, 0)$

Vertices are $(0, -a)$ and $(0, a)$

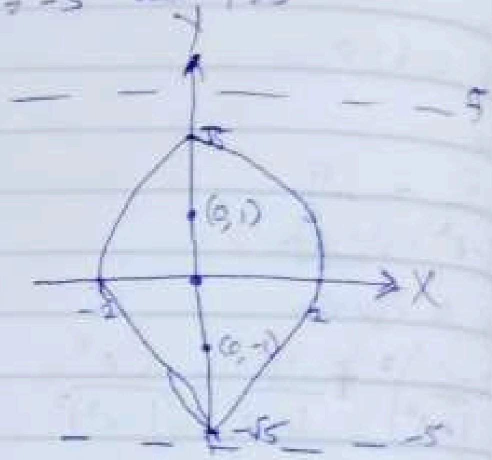
$\Rightarrow$ we have $(0, -\sqrt{5})$ and $(0, \sqrt{5})$

Foci are $(0, -ae)$ and $(0, ae)$

$\Rightarrow (0, -1)$ and $(0, 1)$

Directrices are $y = -\frac{a}{e}$ and $y = \frac{a}{e}$

$y = -5$ and $y = 5$



* The major axis is the y axis because we switched a^2 and b^2

Working in the Original Coordinates for an Ellipse

Example: Obtain the centre, Vertices, foci and directrices of the Conic section

$$5x^2 + 9y^2 - 20x + 36y + 11 = 0$$

Sketch the Conic

Solution

$$5x^2 + 9y^2 - 20x + 36y + 11 = 0$$

$$5x^2 - 20x + 9y^2 + 36y + 11 = 0$$

$$5(x^2 - 4x) + 9(y^2 + 4y) + 11 = 0$$

Using completely the square

$$\Rightarrow 5[(x-2)^2 - 4] + 9[(y+2)^2 - 4] + 11 = 0$$

$$5(x-2)^2 - 20 + 9(y+2)^2 - 36 + 11 = 0$$

$$\Rightarrow 5(x-2)^2 + 9(y+2)^2 = 45$$

$$\Rightarrow \frac{x-2}{9} + \frac{(y+2)^2}{5} = 1$$

Let $X = x-2$, $Y = y+2$

then $\frac{X^2}{9} + \frac{Y^2}{5} = 1$

$\Rightarrow a^2 = 9$; $b^2 = 5$

$a = 3$, $b = \sqrt{5}$

Recall; $b^2 = a^2(1-e^2)$

$\Rightarrow 5 = 9(1-e^2)$

$\Rightarrow e = \frac{2}{3}$

Center in XY Coordinate is $(0, 0)$

We note; $x = X+2$ and $y = Y-2$

$x = 2$, and $y = -2$

thus the center in xy coordinate is $(2, -2)$

Vertices in XY coordinate are $(-a, 0)$ and $(a, 0)$

$\Rightarrow$ ~~$(0, -3)$ and $(0, 3)$~~

$\Rightarrow (-3, 0)$ $(3, 0)$

in the xy coordinates, the vertices are

$x = -3+2 = -1$

and $x = 3+2 = 5$

$y = 0-2 = -2$

and $y = 0-2 = -2$

$\therefore$ in the xy coordinate, the vertices are $(-1, -2)$ and $(5, -2)$

$ae = 3 \times \frac{2}{3} = 2$, $\frac{a}{e} = 3 \times \frac{3}{2} = \frac{9}{2}$

foci is $(-ae, 0)$ and $(ae, 0)$

$\Rightarrow (-2, 0)$ and $(2, 0)$

$x = X+2$, $x = -2+2 = 0$ and $2+2 = 0$

$y = Y-2 \Rightarrow 0-2 = -2$

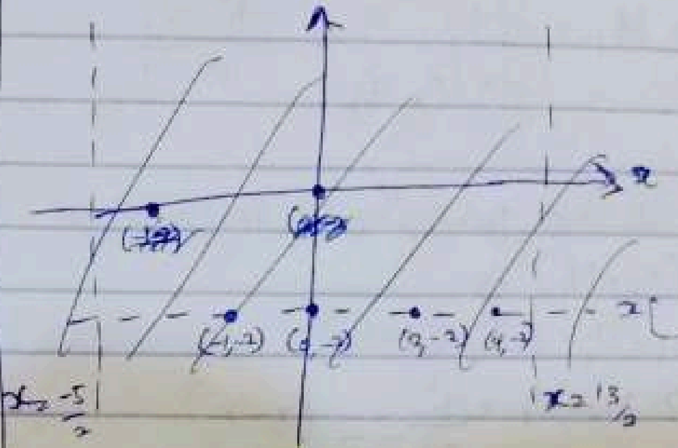
~~focus~~ foci for xy-coordinates

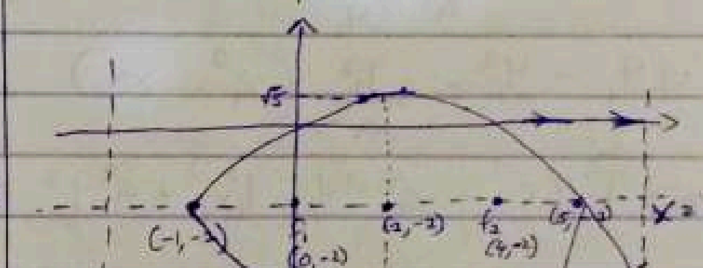
$(0, -2)$ and $(4, -2)$

Directrices in XY-coordinates, $X = \frac{-9}{2}$ and $\frac{9}{2}$

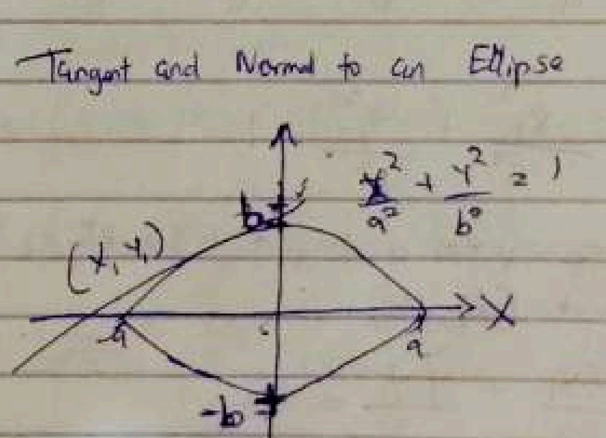
In xy-coordinate, directrices are

$x = \frac{-5}{2}$ and $x = \frac{13}{2}$





Tangent and Normal to an Ellipse



Suppose the line l is a tangent to the ellipse $\frac{X^2}{a^2} + \frac{Y^2}{b^2} = 1$ at (X_1, Y_1)

If M_1 is the gradient of the tangent at (X_1, Y_1) then

$$M_1 = \left.\frac{dY}{dX}\right|_{(X_1, Y_1)}$$

Clearly,

$$\frac{d}{dx}\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)=0 \Rightarrow \frac{2x}{a^2}+\frac{2y}{b^2}\frac{dy}{dx}=0$$

$$\Rightarrow M_1 = \left.\frac{dy}{dx}\right|_{(x_1, y_1)} = \frac{-b^2}{a^2}\frac{x_1}{y_1}$$

Recall that the equation of the tangent at (x_1, y_1) is

$$y - y_1 = M_1(x - x_1)$$

$$\Rightarrow y - y_1 = \frac{-b^2}{a^2}\frac{x_1}{y_1}(x - x_1)$$

$$yy_1 - y_1^2 = \frac{-b^2}{a^2}(x_1^2 - xx_1)$$

$$\Rightarrow b^2 xx_1 + a^2 yy_1 = b^2x_1^2 + a^2y_1^2$$

$$\Rightarrow \frac{xx_1}{a^2} + \frac{yy_1}{b^2} = \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2}$$

But $\dfrac{x_1^2}{a^2} + \dfrac{y_1^2}{b^2} = 1$

then the equation of tangent to the ellipse at point (x_1, y_1) is

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$$

Exercise

Obtain the equation of normal to the curve (ellipse) at point (x_1, y_1)

Soln

Equation of the normal can be obtained from the fact that

$$M_2 = \frac{-1}{M_1} = \frac{a^2}{b^2}\frac{y_1}{x_1}$$

where M_2 is d gradient of normal at (x_1, y_1)

Equn of normal is

$$y - y_1 = \frac{a^2}{b^2}\frac{y_1}{x_1}(x - x_1)$$

$$b^2x_1y - b^2x_1y_1 = a^2xy_1 - a^2x_1y_1$$

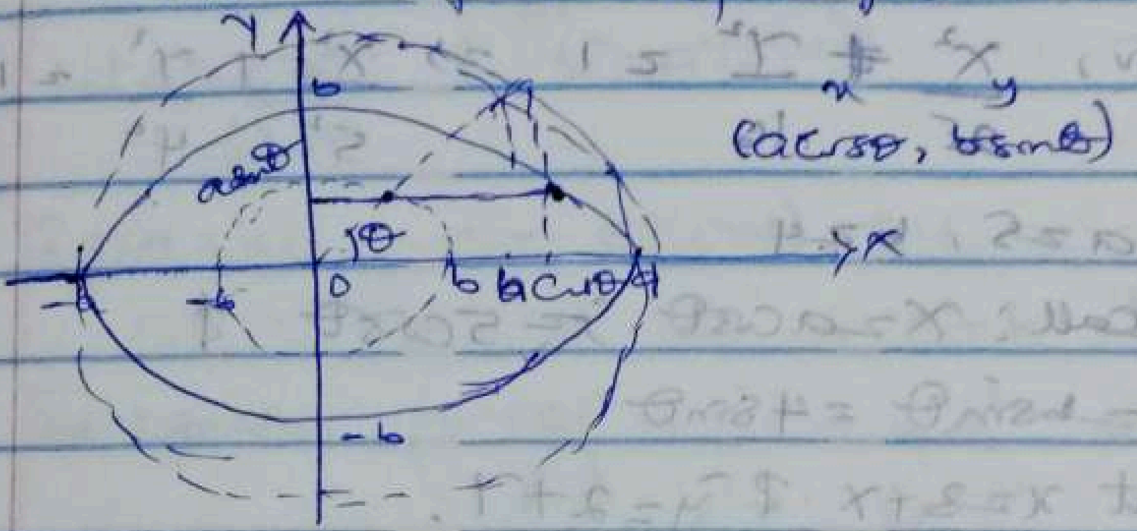
$$a^2xy_1 - b^2x_1y = a^2x_1y_1 - b^2x_1y_1$$

$$\frac{xy_1}{b^2} - \frac{x_1y}{a^2} = \frac{x_1y_1}{b^2} - \frac{x_1y_1}{a^2}$$

$$\frac{x_1y}{a^2} - \frac{xy_1}{b^2} = \frac{x_1y_1}{a^2} - \frac{x_1y_1}{b^2}$$

$\cancel{\frac{1}{a^2}\left(\frac{xx_1}{a^2} - \frac{yy_1}{b^2}\right)}$

Parametric Equations of Ellipse



From the figure, the point P has the coordinates $X = a\cos\theta$, $Y = b\sin\theta$.

Since P is an arbitrary point on the ellipse, the above eqns are called parametric eqn of an ellipse.

The proof is as follows:

$X^2 = a^2\cos^2\theta$ & $Y^2 = b^2\sin^2\theta$

$$\Rightarrow \frac{X^2}{a^2} + \frac{Y^2}{b^2} = \cos^2\theta + \sin^2\theta \Rightarrow \frac{X^2}{a^2} + \frac{Y^2}{b^2} = 1$$

Cartesian Eqn of an ellipse

Example: Obtain parametric equation of the ellipse.

$16(x-3)^2 + 25(y-2)^2 = 400$

Soln

The ellipse can be written as:

$$\frac{(x-3)^2}{25} + \frac{(y-2)^2}{16} = 1 \quad \text{[Divide thru by 400]}$$

Let $X = x-3$ and $Y = y-2$

So, $\frac{X^2}{25} + \frac{Y^2}{16} = 1 \Rightarrow \frac{X^2}{5^2} + \frac{Y^2}{4^2} = 1$

$\therefore a = 5,\ b = 4$

Recall: $X = a\cos\theta = 5\cos\theta$ &

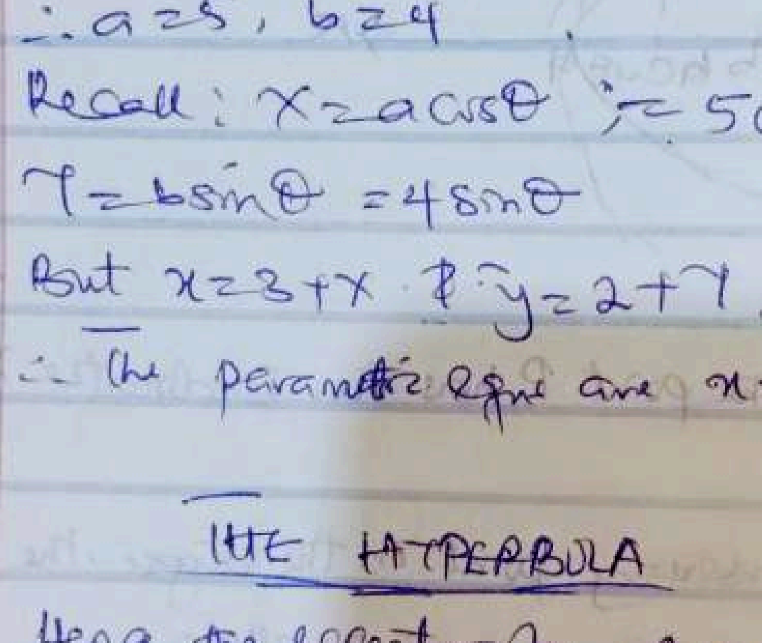
$Y = b\sin\theta = 4\sin\theta$

But $x = 3 + X$ & $y = 2 + Y$.

$\therefore$ the parametric eqns are $x = 3 + 5\cos\theta$ & $y = 2 + 4\sin\theta$

THE HYPERBOLA

Here the eccentricity $e > 1$. Consider the figure:



Recall:

$|PF|^2 = e^2|PK|^2$

$(x-c)^2 + y^2 = e^2x^2$

$0 = (e^2-1)x^2 + 2cx - c^2 - y^2$

$(e^2-1)x^2 + 2cx - y^2 = c^2$

$\frac{\cancel{e^2-1}}{\cancel{e^2-1}}x^2 + \frac{2cx}{e^2-1} - \frac{y^2}{e^2-1} = \frac{c^2}{e^2-1}$

$$\left(x+\frac{c}{e^2-1}\right)^2-\frac{y^2}{e^2-1}=\frac{e^2}{e^2-1}+\frac{c^2}{(e^2-1)^2}$$

$$\left(\frac{x+c}{e^2-1}\right)^2-\frac{y^2}{e^2-1}=\frac{c^2e^2}{(e^2-1)^2}$$

Let $a^2=\frac{c^2e^2}{(e^2-1)^2}$ & $b^2=a^2(e^2-1)$

If $a,b>0$, $c=\frac{a(e^2-1)}{e}$ & $\frac{1}{e^2-1}=\frac{a^2}{b^2}$

The equation: $\left(x+\frac{c}{e^2-1}\right)^2-\frac{y^2}{e^2-1}=\frac{c^2e^2}{(e^2-1)^2}$ can be written as

$$\left(x+\frac{a}{e}\right)^2-\frac{a^2}{b^2}y^2=a^2 \Rightarrow \frac{\left(x+\frac{a}{e}\right)^2}{a^2}-\frac{y^2}{b^2}=1$$

Let $X=x+\frac{a}{e}$ & $Y=y$

The standard form a hyperbola is given as:

$$\frac{X^2}{a^2}-\frac{Y^2}{b^2}=1$$

For the vertex, we note that the hyperbola intersects the X-axis at $(-a,0)$ & $(a,0)$.

The centre of the hyperbola is $(0,0)$.

foci is obtained from the fact that in xy-coordinate the focus is at $(c,0)$ & by $X=x+\frac{a}{e}$, we've

$X=c+\frac{a}{e}=ae$

Hence, the foci in XY-coordinates are $(-ae, 0)$ & $(ae, 0)$

In the xy-coordinate, the directrix is $x=0$ and $X = x+3$, the directrices are:

$$X = -\frac{a}{e} \quad \& \quad X = a/e$$

Asymptotes of Hyperbola

The hyperbola can be written as

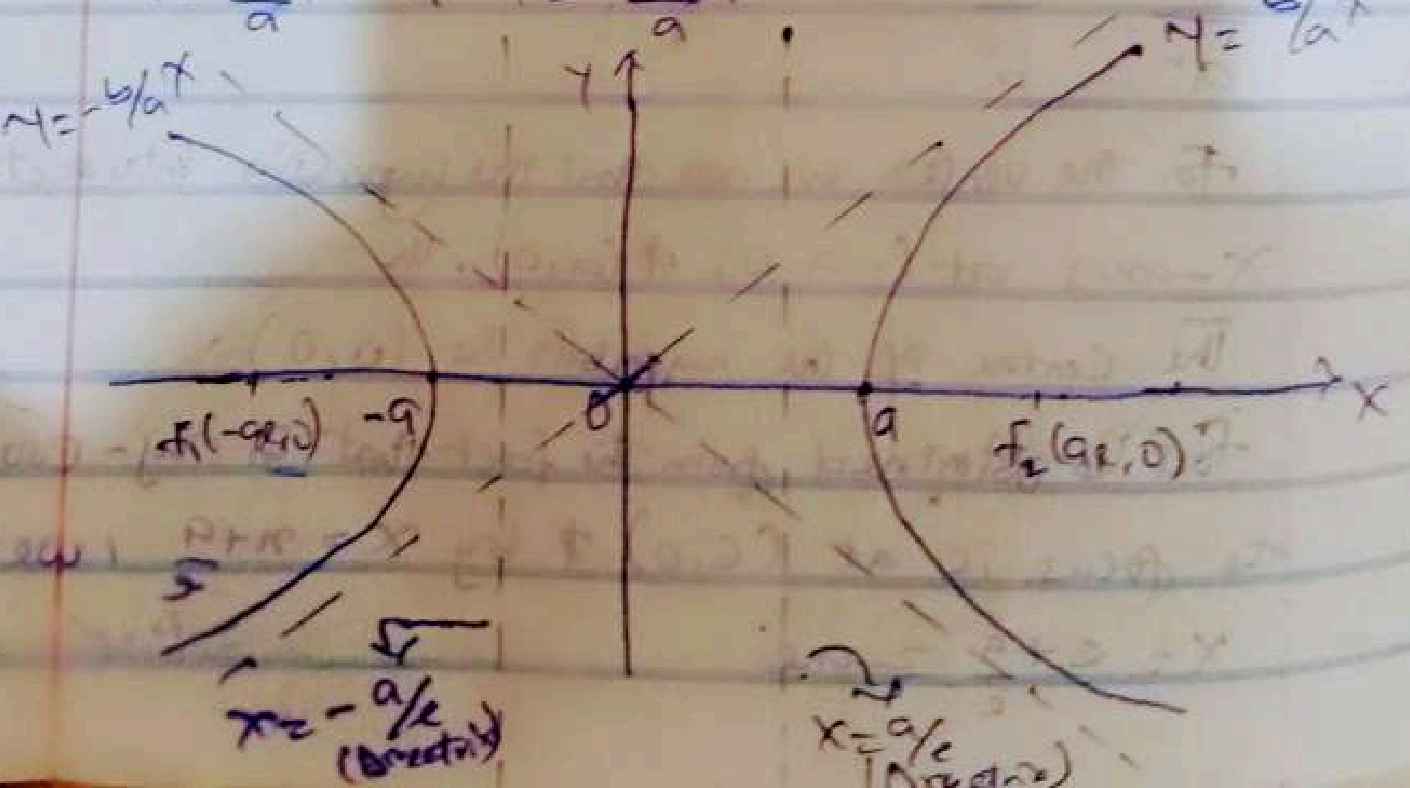
$$\frac{Y^2}{b^2} - \frac{X^2}{a^2} = 1$$

Clearly, when x gives a very large value, we've

$$\frac{Y^2}{b^2} \simeq \frac{X^2}{a^2}$$

The asymptotes of the hyperbola are therefore given

$$Y = \frac{b}{a}X \quad \& \quad Y = -\frac{b}{a}X$$



Example) Obtain the centre, vertices, focii, directrices & asymptotes of the hyperbolas (i) $\frac{x^2}{4} - \frac{y^2}{9} = 1$ (ii) ...

Soln

(i) $\frac{x^2}{4} - \frac{y^2}{9} = 1$, centre $= (0,0)$, $a^2 = 4$, $a = 2$, $b^2 = 9$, $b = 3$

from; $b^2 = a^2(e^2 - 1) \Rightarrow 9 = 4(e^2 - 1) \Rightarrow e^2 = 13/4$, $e = \frac{\sqrt{13}}{2}$

vertices are $(-a, 0)$ & $(a, 0)$

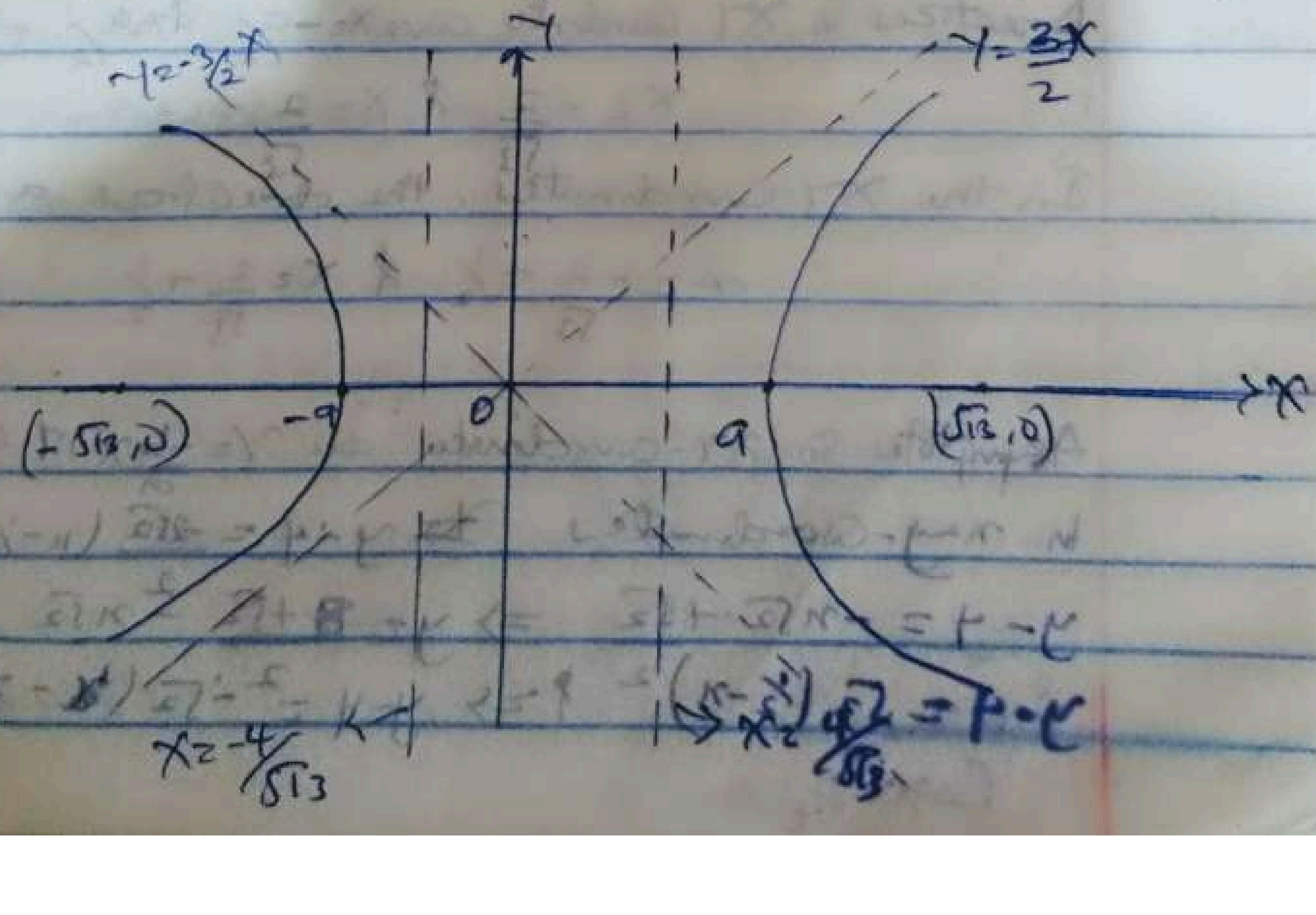
$= (-2, 0)$ & $(2, 0)$

$ae = \frac{\sqrt{13}}{2} \times 2 = \sqrt{13}$

focii are $(-ae, 0)$ & $(ae, 0) \Rightarrow (-\sqrt{13}, 0)$ & $(\sqrt{13}, 0)$

Directrices are $x = -\frac{a}{e}$ & $x = \frac{a}{e} \Rightarrow x = -\frac{4}{\sqrt{13}}$ & $x = \frac{4}{\sqrt{13}}$

Asymptotes are; $y = -\frac{b}{a}x$ & $y = \frac{b}{a}x \Rightarrow y = -\frac{3}{2}x$ & $y = \frac{3}{2}x$



Working in the Original Coordinates for Hyperbola

Examples: Obtain the foci, directrices & eqn^s of the asymptotes of the hyperbola:

$$\frac{(x-\frac{1}{2})^2}{4} - \frac{(y-4)^2}{8} = 1 \quad \text{Let } X = x - \tfrac{1}{2} \;\&\; Y = y-4$$

i.e. $x = X + \frac{1}{2}$ & $y = Y+4$

Clearly, the hyperbola can be written as

$$\frac{X^2}{4} - \frac{Y^2}{8} = 1, \quad a^2 = 4,\ b^2 = 8,\ a = 2,\ b = 2\sqrt{2}$$

Recall; $b^2 = a^2(e^2-1)$

$ae = 2\sqrt{3}$ $\qquad 8 = 4(e^2-1)$, $e^2 = 3$, $e = \sqrt{3}$

foci in XY coordinates are $(-ae, 0)$ & $(ae, 0)$

$= (-2\sqrt{3}, 0)$ & $(2\sqrt{3}, 0)$

In xy-coordinates, the foci are $(-2\sqrt{3}+\frac{1}{2}, 0)$ & $(2\sqrt{3}...$

Directrices in XY coordinates are $X = -\frac{a}{e}$ & $X = \frac{a}{e}$ where $\frac{a}{e} = \frac{2}{\sqrt{3}}$

$$X = -\frac{2}{\sqrt{3}} \;\&\; X = \frac{2}{\sqrt{3}}$$

In the XY coordinates, the directrices are

$$x = \frac{-2}{\sqrt{3}} + \frac{1}{2} \;\&\; x = \frac{2}{\sqrt{3}} + \frac{1}{2}$$

Asymptotes in XY-coordinates are $Y = \frac{-b}{a}X$ & $Y = \frac{b}{a}X$

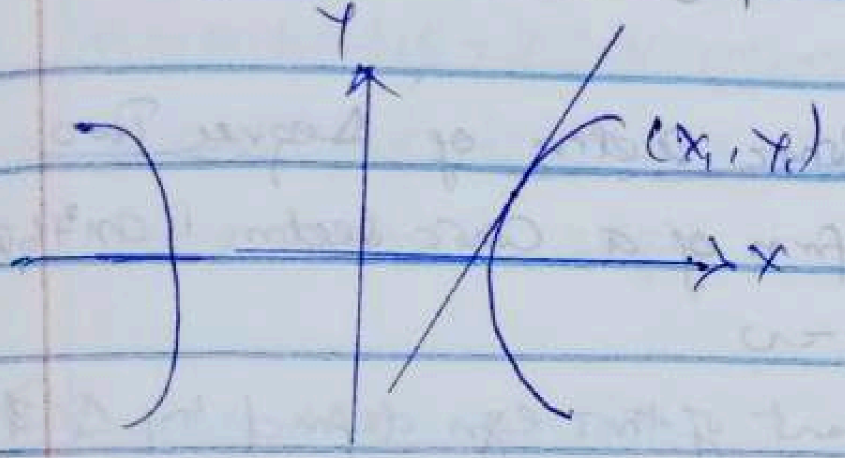
In xy-coordinates $\Rightarrow y - 4 = \frac{-2\sqrt{2}}{2}(x - \frac{1}{2}) \Rightarrow$

$y - 4 = -x\sqrt{2} + \frac{\sqrt{2}}{2} \Rightarrow y = \frac{8+\sqrt{2}}{2} - x\sqrt{2}$ &

$y - 4 = \sqrt{2}(x - \frac{1}{2}) \Rightarrow y - 4 = \sqrt{2}(x - \frac{1}{2})$

$y-4 = \sqrt{2}\,(x-\tfrac{1}{2})$

Tangent & Normal to Hyperbola



- Following the pattern for Parabola & ellipse, the eqn of tangent to a hyperbola at point (x_1, y_1) is given as

$$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$$

Parametric Equations of Hyperbola

Parametric eqns of a parabola, $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, $x = a\sec\theta$, $y = b\tan\theta$

Observe that, $x^2 = a^2\sec^2\theta$, $y^2 = b^2\tan^2\theta$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = \sec^2\theta - \tan^2\theta \Rightarrow \underbrace{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1}_{\text{Cartesian figure}}$$

Example: Obtain the parametric equations of the Conic section

$4x^2 - 9y^2 - 8x + 36y - 113 = 0$

Clearly,

$4x^2 - 8x - 9y^2 + 36y - 113 = 0$

$4(x^2 - 2x) - 9(y^2 - 4y) - 113 = 0 \Rightarrow 4[(x-1)^2 - 1] - 9[(y-2)^2 - 4) - 113 = 0$

$4(x-1)^2 + 9(y-2)^2 = 81 \Rightarrow \frac{(x-9)^2}{(9/2)^2} - \frac{(y-2)^2}{3^2} = 1$

$X = x-1,\ Y = y-2 \Rightarrow \frac{X^2}{(9/2)^2} - \frac{Y^2}{3^2} = 1$

$\Rightarrow X^2$

Recall; $x = a\sec\theta$ & $y = b\tan\theta \Rightarrow x_1 = \frac{9}{2}\sec\theta$. $y_1 =$

The parametric eqns are

$x = \frac{9}{2}\sec\theta$ ~~...~~ $y = 2 + 3\tan\theta$

General Conic Section of Degree Two

The general form of a conic section is $ax^2 + bxy + cy^2 +$ $2gx + 2fy + d = 0$

The discriminant of this eqn defined by Δ & defined as $\Delta = b^2 - 4ac$.

from this, we've

(i) If $\Delta = 0$, then the conic section is a parabola

(ii) If $\Delta < 0$, then the conic section is an ellipse

(iii) If $\Delta > 0$, then " " " " a hyperbola

Example:- Determine the type of conic section of the following: (i) $x^2 + 2xy + y^2 - 2x + 4y - 5 = 0$

(ii) $x^2 + xy + y^2 - x - 4y = 5$

(iii) $2xy - y + x = 6$

Soln

(i) $x^2 + 2xy + y^2 - 2x + 4y - 5 = 0$

$a = 1, b = 2, c = 1$

Recall: $\Delta = b^2 - 4ac = 4 - 4 = 0$

Conic section is

Conic section is a parabola

(ii) $x^2 - xy + y^2 - x - 4y = 5$

$a = 1, b = 1, c = 1$

$\Delta = b^2 - 4ac = 1 - 4 = -3 < 0$

$\therefore$ Conic section is ellipse

(iii) $xy - y + x = 6$

$a = 0, b = 1, c = 1$

$\Delta = b^2 - 4ac = 1 > 0$

$\therefore$ Conic section is hyperbola

STATISTICS

This is a branch of science dealing with the collection of data, organization, summarization, presentation, analysis of data and drawing valid conclusions.
Thereafter, making reasonable decisions on the basis of such analyses.

Frequency Distribution

This is the arranged data which are summarized distributing them into classes or categories with their frequencies. The frequency distributions can either be grouped or ungrouped.

Example 1: A typical grouped frequency table is given on table 1.

Table 1: Wages of 100 workers

Wages in Naira	0-10	10-20	20-30	30-40	40-50
Number of workers	12	23	35	20	10

While for ungrouped, we have Table 2

Table 2: Test scores of 35 students.

Mark (x)	0	1	2	3	4	5	6	7	8	9	10
No of students with the mark (x)		2	2	0	5	5	8	5	3	2	1

With the frequency distribution of data set in place, different diagrams such as Histogram, frequency polygon, frequency curve, Cummulative frequency curve (OGIVE) Bar chart, Circles (Pie chart) are often useful to represent these distribution.

Graphical Representation of frequency Distributions

A useful definition which is pertinent to this subject matter is the definition of class frequency/width.

CLASS INTERVAL / WIDTH [DENOTED BY C]

This is the difference between the upper-class limit and the lower-class limit. It refers to the numerical width of any class in a particular distribution.

1. HISTOGRAM: There are two types of histogram, namely: Histogram with equal class-interval and Histogram with unequal class-intervals.

Histogram with equal class-interval consists of a set of rectangles having their Height proportional to the class-frequencies while histogram with unequal class-interval have the areas of the rectangles proportional to the frequencies. Interest here is ~~the~~ limited to the Histogram with equal class-interval.

Hence, histogram is a plot of class-frequencies plotted against class-intervals. Although class-intervals are used but class boundaries are preferred in practice and they are obtained by the following step:

(a) Substract the upper limit for the 1st class from the lower class limit for the 2nd class.

(b) Divide the result in (a) by 2

(c) Substract the result from the 1st lower class limit and add it to all ~~of~~ the upper class limits.

(d) For the 2nd lower class limit, 3rd class limit, etc repeat the upper class limit values in the class just before.

Example 3:- Obtain the class boundaries for the data on Table 3.

Table 3: Heights of 39 males.

Height (m)	60-63.9	64-67.9	68-71.9	72-75.9	76-79.9
frequency	4	9	15	8	3

Soln

For step a: $64 - 63.9 = 0.1$

For step b: $\frac{0.1}{2} = 0.05$

for step c: $60.00 - 0.05 = 59.95$ and $63.9 + 0.05 = 63.95$

67.9+0.05 = 67.95   75.9+0.05 = 75.95

71.9+0.05 = 71.95   79.9+0.05 = 79.95

Hence, the resulting class boundaries for Table 3 can be seen on Table 4 as follows:

Table 4: Heights of 39 males

Boundaries

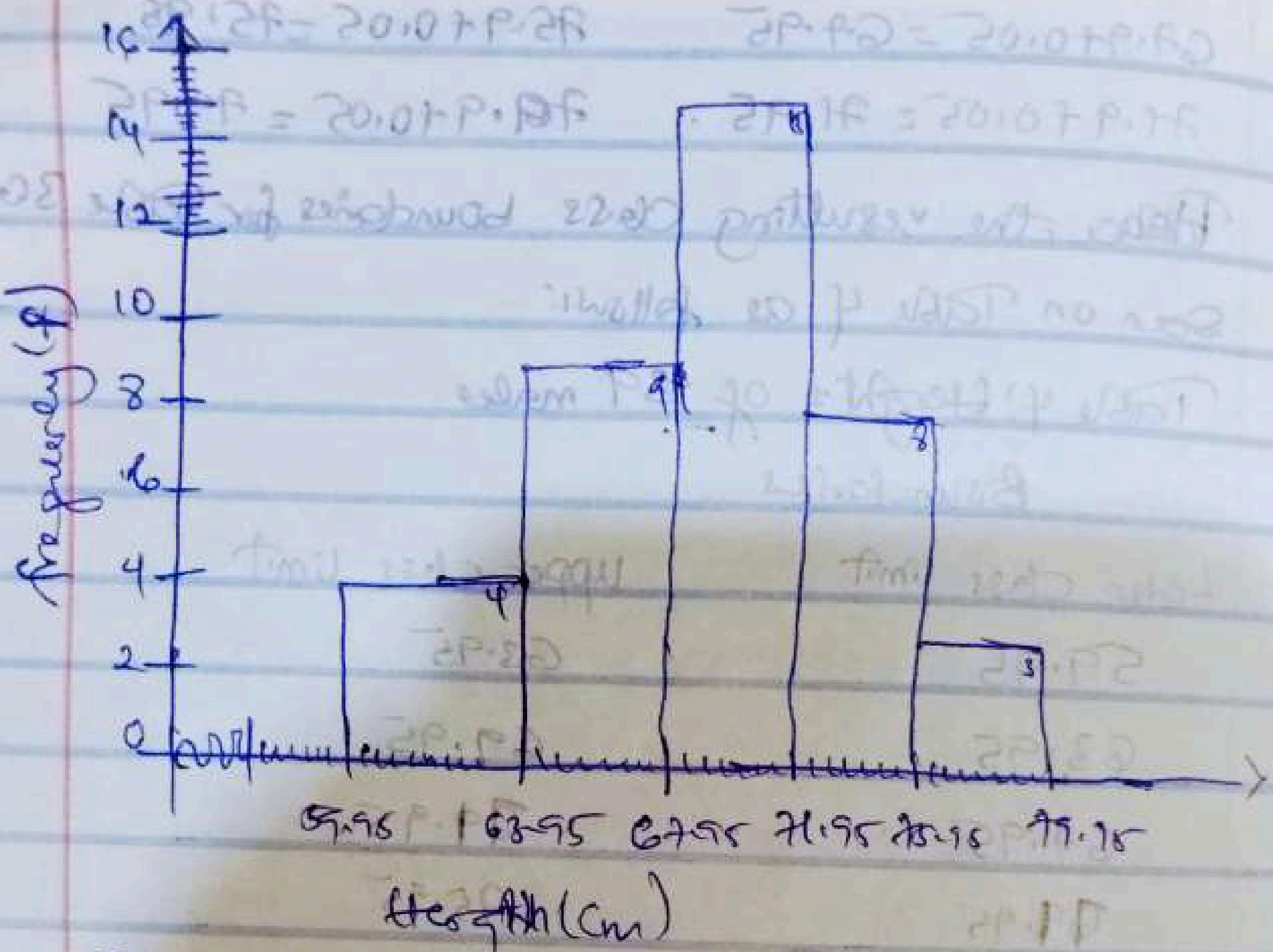
Lower class limit	Upper class limit
59.95	63.95
63.95	69.95.
67.95	71.95
71.95	75.95
75.95	79.95

Note here that the essence of finding class boundaries is to achieve overlapping classes - this implies having the upper limit of a class coinciding with the lower limit of the class just immediately after it, see the constructed histogram in figure 1.

Height (cm)	59.95-63.95	63.95-69.95	67.95-71.95	71.95-75.95	75.95-79.95
freq. (F)	4	9	15	8	3

Scale: 1cm represents 2units on y-axis

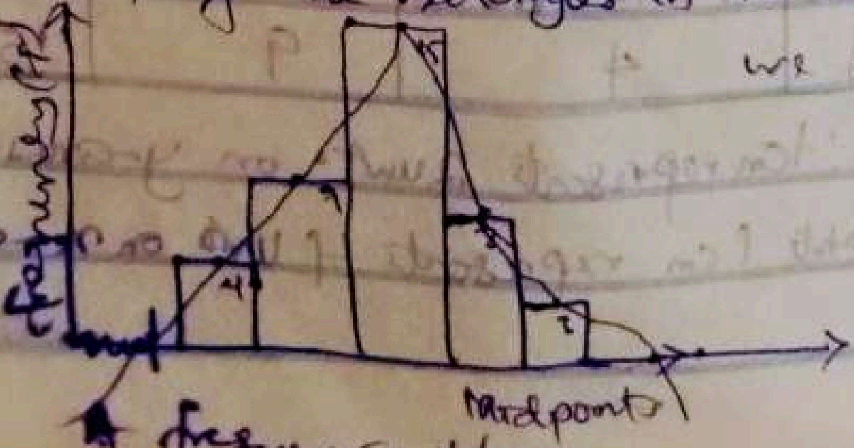
while 1cm represents 4 units on x-axis



~~Figure 4: A~~ ~~Histogram~~ showing the heights of 39 males

② ~~FREQUENCY~~ POLYGON

This is a line graph of class frequency plotted against class mark. It is obtained by connecting mid points on the tops of the rectangles in the histogram. See figure



A frequency Polygon

③ Cummulative frequency Curve (or Ogive)

When the various points are plotted according to the upper boundary of the class as x-coordinate and the Cummulative frequency Curve is obtained by joining these point by a free hand Smooth Curve.

Example 3: The Cummulative frequency Curve corresponding to Table 1 is constructed using the following steps:

STEP 1: Prepare a "less than table" of the lower Class boundaries as follows.

Table 5: A less than table for wages of 100 workers.

Wages	frequency	Cummulative freq
0-10	12	12 (i.e 12 Candidates with wages <10
10-20	23	35 (i.e 35 Candidates with wages <20
20-30	35	70 (i.e (35+35) " " " <30
30-40	20	90 (70+20) " " " <40
40-50	10	100 (90+10) " " " <50
	$\Sigma f = 100$	

Note here that in practise, it is taken that there is a Class having a cummulative freq of 0 for Candidates with wages <0 before 0-10.

STEP 2: Make an abridge table having the columns for the wages and cummulative frequencies against to upper class boundaries to obtain figure 3.

Wages (₦)	0-10	10-20	20-30	30-40	40-50
Cummulative freq	12	35	70	90	100

Scale: 1cm rep. 10 units on both x & y. axis

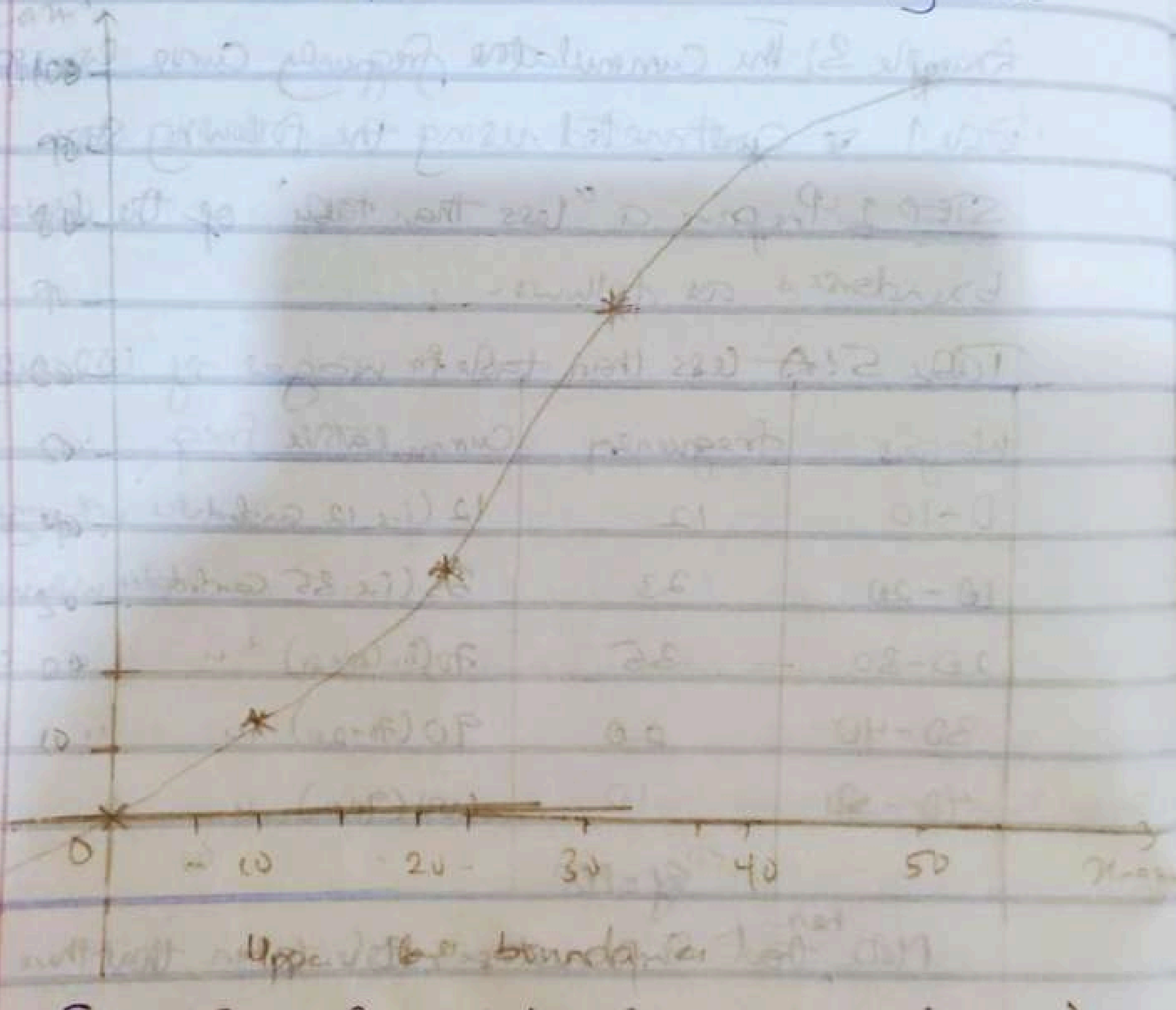


Figure 3: A Cummulative frequency Curve (or Ogive) showing the wages of 100 workers.

AVERAGE OF MEASURE OF CENTRAL TENDENCY

An average is a value which is representative of a set of data. Note here that average value may also be termed as measuring central tendency. There are five types of averages in common, namely:

(a) Arithmetic average or mean

(b) Geometric mean

(c) Harmonic mean

(d) Mode

(e) Median

(a) Arithmetic mean :- If $x_1, x_2, \ldots, x_n$ are n numbers, then their arithmetic mean (A.M) is defined by

$$\bar{x} = \frac{x_1 + x_2 + \cdots + x_n}{n} = \frac{\sum_{i=1}^{n} x_i}{n} \quad \text{—— (1)}$$

If the numbers x_1 occurs f_1 times, x_2 occurs f_2 times and so on, then,

$$\bar{x} = A.M = \frac{f_1 x_1 + f_2 x_2 + \cdots + f_n x_n}{f_1 + f_2 + \cdots + f_n} = \frac{\sum_{i=1}^{n} f_i x_i}{\sum_{i=1}^{n} f_i} \quad \text{—— (2)}$$

Remark

Eqns (1) & (2) are known as direct method. Other alternatives methods are the "shortcut method" and the "step deviation method".

Ex 4 :- Find the mean of 20, 22, 25, 28, 30

Soln

$$\bar{x} = \frac{20+22+25+28+30}{5} = \frac{125}{5} = 25$$

$\bar{x} = 25$

Ex 5 :- Find the mean of the frequency distribution on table 6

Table 6 : A freq distribution table

No	8	10	15	20
freq	5	8	8	4

Soln

$$\bar{x} = \frac{(8\times5)+(10\times8)+(15\times8)+(20\times4)}{(5+8+8+4)} = \frac{40+80+120+80}{}$$

(ii) Short cut method :- Suppose "a" be the assumed mean, "d" the deviation of the variate x from a then

$$\frac{\sum_{i=1}^{n} f_i d_i}{\sum_{i=1}^{n} f_i} = \frac{\sum_{i=1}^{n} f_i(x_i-a)}{\sum_{i=1}^{n} f_i} = \frac{\sum_{i=1}^{n} f_i x_i}{\sum_{i=1}^{n} f_i} - \frac{\sum_{i=1}^{n} f_i a}{\sum_{i=1}^{n} f_i}$$

$$= A.M - \frac{a\sum_{i=1}^{n} f_i}{\sum_{i=1}^{n} f_i} = A.M - a$$

$$\cancel{\Rightarrow} A.M = a + \frac{\sum f_i d_i}{\sum f_i} = A.M - a$$

i.e $\bar{x} = AM = a + \frac{\Sigma fd}{\Sigma f}$ —— (3)

Ex 6: Find the arithmetic mean for the following distribution.

Table 8: A frequency distribution table

Class	0-10	10-20	20-30	30-40	40-50
freq	7	8	20	10	5

Soln: Using eq (3), we construct table 9 as follows

Table 9: Computation of the arithmetic mean of a grouped frequency distribution table using the short cut method

Class	freq	x	d = x - a	fd
0-10	7	$\frac{0+10}{2} = 5$	5-25 = -20	7 × -20 = -140
10-20	8	$\frac{10+20}{2} = 15$	15-25 = -10	8 × -10 = -80
20-30	20	$\frac{20+30}{2} = 25 = a$	25-25 = 0	20 × 0 = 0
30-40	10	$\frac{30+40}{2} = 35$	35-25 = 10	10 × 10 = 100
40-50	5 Σf = 50	$\frac{40+50}{2} = 45$	45-25 = 20	5 × 20 = 100, Σfd = -20

Recall eq (3) we've, $\bar{x} = a + \frac{\Sigma fd}{\Sigma f}$

$\therefore \bar{x} = 25 + \left(\frac{-20}{50}\right) = 25 - \frac{2}{5} = \frac{123}{5} = 24.6$

(iii) Step Deviation Method: Let "a" be the assumed mean, "c" the width of the class interval and "u" $= \frac{d}{c} = \frac{x-a}{c}$, then

$\bar{x} = A.M = a + \frac{c\Sigma fu}{\Sigma f}$ —— (4)

Ex 7: Find the arithmetic mean for the data on table 10

Table 10: A grouped frequency distribution table

Class	0-10	10-20	20-30	30-40	40-50
freq	7	8	20	10	5

Soln

$C = (10-0) = (20-10) = (30-20) = (40-30) = (50-40) = 10$

Now using eq (4) we construct Table 11 as follows:

Table 11: Computation of the arithmetic mean of a grouped frequency distribution table using the step deviation method.

Class	Midpoint (x)	freq	x-a	$u = \frac{x-a}{c}$	fu
0-10	$\frac{10+0}{2} = 5$	7	5-25=-20	$-20/10 = -2$	7x-2 = -14
10-20	$\frac{10+20}{2} = 15$	8	15-25=-10	$-10/10 = -1$	8x-1 = -8
20-30	$\frac{20+30}{2} = 25 = a$	20	25-25=0	$0/10 = 0$	20x0 = 0
30-40	$\frac{30+40}{2} = 35$	10	35-25=10	$10/10 = 1$	10x1 = 10
40-50	$\frac{40+50}{2} = 45$	5	45-25=20	$20/10 = 2$	5x2 = 10
		$\Sigma f = 50$			$\Sigma fu = -2$

$$\therefore \bar{X} = A.M = 25 + \left[10 \times \left(\frac{-2}{50}\right)\right] = 25 - \frac{2}{5} = \frac{123}{5} = 24.6$$

which is the same answer as using the short-cut method.

$\frac{-x}{5} = \frac{p}{5} = $ "w" bro

(P.)

(5) Geometric mean :- If $x_1, x_2, \dots, x_n$ be n values of variate x, then the geometric mean

$$G = G.M = (x_1 * x_2 \times \dots \times x_n)^{1/n} \quad \text{--- (5)}$$

Ex 8 :- find the geometric mean of 4, 8, 16.

Soln

$x_1 = 4$, $x_2 = 8$ & $x_3 = 16$, => $n = 3$

$$G = G.M = (4 \times 8 \times 16)^{1/3} = (512)^{1/3} = \sqrt[3]{512} = 8$$

(6) Harmonic mean :- Harmonic mean of a series of values is defined as the reciprocal of the arithmetic mean of their reciprocals. Thus, If H be the harmonic mean, then

$$\frac{1}{H} = \frac{1}{HM} = \frac{1}{n}\left[\frac{1}{x_1} + \frac{1}{x_2} + \dots + \frac{1}{x_n}\right] \quad \text{--- (6)}$$

Ex 9 :- Calculate the harmonic mean of 4, 8, 16

Soln : $x_1 = 4$, $x_2 = 8$, $x_3 = 16$, $n = 3$

but $\frac{1}{H} = \frac{1}{HM} = \frac{1}{3}\left(\frac{1}{4} + \frac{1}{8} + \frac{1}{16}\right) = \frac{7}{48}$

Upon inversion, we've that

$$H = H.M = \frac{48}{7} = 6.853$$

REMARKS

(1) $H.M \leq G.M \leq A.M$

(2) Equality sign holds only if all sample values are identical.

Ans. (a) Mode : Mode is defined to be the size of the variable which occurs most frequently. The value of this measure can be obtained for both ungrouped & grouped frequency distribution data sets.

Ex 10 : Find the mode of the following items

0, 1, 6, 7, 2, 3, 7, 6, 6; 2, 6, 0, 5, 6, 0

Sln : The mode of 15 items

Items	Tally	Frequencies
0	111	3
1	1	1
2	11	2
3	1	1
5	1	1
6	++++	$\boxed{5}$ = MODE
7	11	2

For grouped data, can be determined by

(i) Using the formula,

$$MODE = L_1 + \left(\frac{D_1}{D_1 + D_2}\right) C$$

where L_1 = Lower class boundary of the modal class

D_1 = Modal class frequency minus frequency of the next lower class

D_2 = Modal class frequency minus frequency of the next higher class

C = Modal class size

(ii) Directly from the histogram, using example 11 to illustrate both methods, we've

Ex 11:- Find the mode from the following data

Table 13: Mode of a grouped frequency distribution

Items	frequencies
0 - 6	6
6 - 12	4
12 - 18	25
18 - 24	35
24 - 30	18
30 - 36	12
36 - 42	6

Solution using the formula method

Table 14: Mode of grouped freq. distribution table

Items	frequencies
0 - 6	6
6 - 12	4
12 - 18	25
$\boxed{18 - 24}$ modal class	$\boxed{35}$ modal class freq.
24 - 30	18
30 - 36	12
36 - 42	6

We've from table 14 that the modal class = 18-24. Since there's coincidence of the upper limit of one class with the lower limit of the class immediately to it, then there's no need to find class boundaries. The grouping as they appear are used as class boundaries.

$\Rightarrow L_1 = 18, \quad \Delta_1 = 35 - 25 = 10, \quad \Delta_2 = 35 - 18 = 17$

$C = (6-0) = (12-6) = (18-12) = \cdots = (42-36) = 6$

Using eq (7), we've

of MODE $= 18 + \left(\frac{10}{10+17}\right)6 = 18 + \frac{10}{27} \times 6 = \frac{182}{9} \approx 20.22$

≈ 20.22

Solution using the Histogram :- You need to construct the corresponding histogram; spot the highest bar and do these on the highest bar afterward (see fig 4 for details)

Items	0-6	6-12	12-18	18-24	24-30	30-36	36-42
freq.	6	11	25	35	18	12	6

Scale 1 cm rep. 6 units on x-axis while 1cm rep. 10 units on y-axis

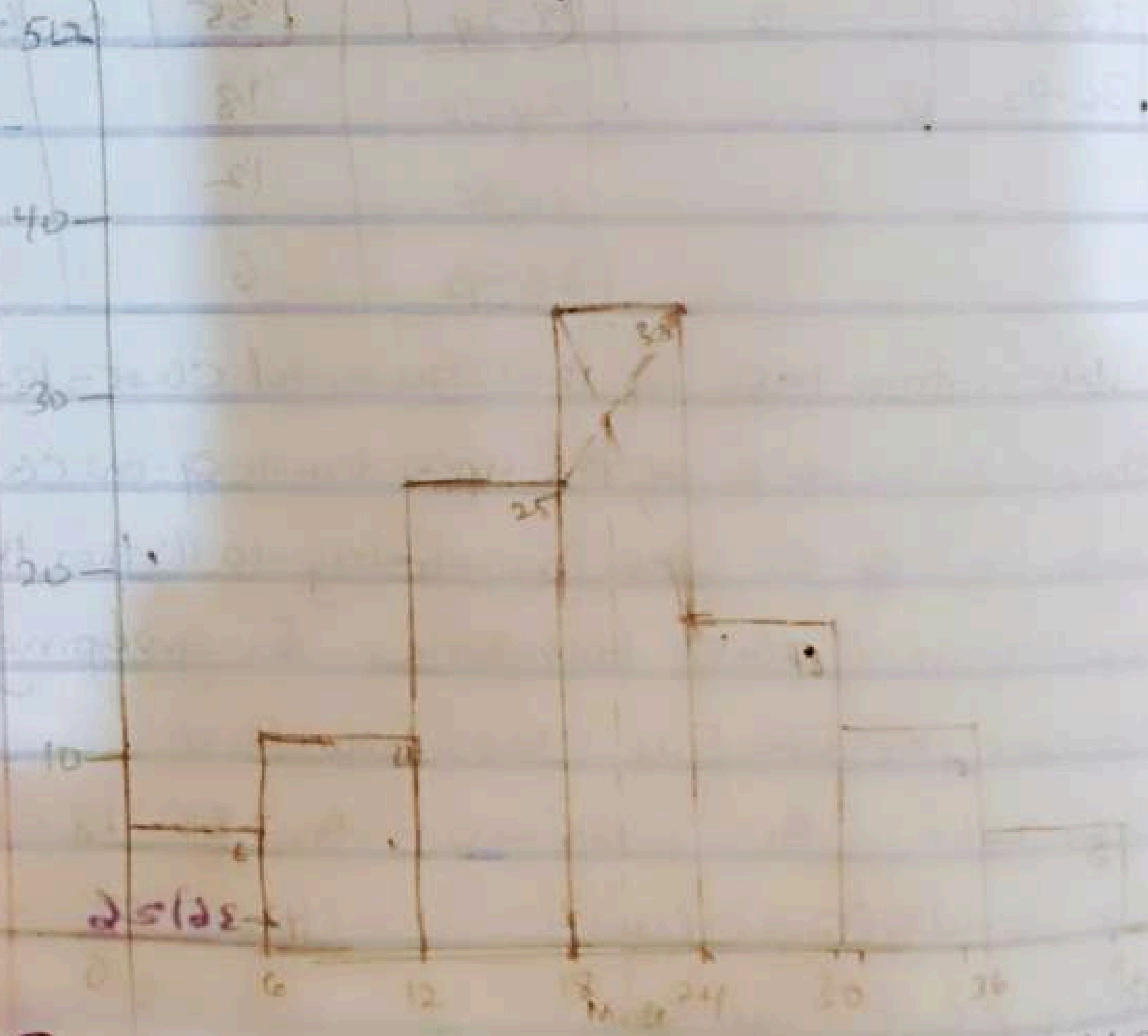


Figure 4) A histogram indicating the mode within a grouped freq. distribution

modal class of a

(2) Median: Median is defined as the measure of the central item when they're arranged in ascending or descending order of magnitude.

The total number of the item (frequency) may be odd or even. When the total number of the item is odd and equal to say n, the the $\left[\frac{1}{2}(n+1)\right]$ item gives the median while when the total number of items is even, say n, then there are two value of $\left[\frac{1}{2}(n)^{th} \text{ and } \frac{1}{2}(n+1)^{th}\right]$ items, is the median. But like other measures of central tendencies (the mean & mode) the median can be determined for both ungrouped & grouped data.

For ungrouped data, we've

Eg 12: Find the median of 6, 8, 9, 10, 11, 12, 13.

Sln: Total no of items (i.e sample size) = 7, => $n = 7$ which is odd

$\therefore$ Median $= \left[\frac{1}{2}(n+1)^{th}\right]$ item $= \left(\frac{1}{2}(7+1)\right)^{th} = \left(\frac{1}{2} \times 8\right)^{th} = 4th$

$= 10$

For grouped data, we've that the $\left[\frac{1}{2}(N+1)^{th}\right]$ number of the distribution gives the median & this median value can be obtained using 3 methods, namely:

(1) METHOD BY CALCULATION USING THE FORMULA

$$\text{Median} = L_1 + \left(\frac{N/2 - (\Sigma f)_L}{f \text{ median}}\right) C$$

where L_1 = Lower class boundary of the median class

f median = frequency of the median class

C = Class size

$(\Sigma f)_2$ = sum of frequencies of all classes lower than the median class

$N = \Sigma f$ = Total frequency of the data.

(i) METHOD BY CALCULATION USING THE FORMULA

$$\text{Median} = L_1 + \left(\frac{P_1 C}{f_{median}}\right)$$ where,

P_1 = Position of the median in the median class containing the median

L_1 = Lower class boundary of the median class

C = Class size

f_{median} = frequency of the median class.

(ii) USING A CUMMULATIVE FREQUENCY CURVE (OGIVE) TO DETERMINE THE MEDIAN.

Recall the cummulative frequency curve (or ogive) is a plot of cummulative frequencies against the class boundaries of a frequency distribution

Ex 13: Suppose a sample of 200 spanners is taken from the daily production in a factory, and the distance between the faces measured to the nearest 0.1mm Suppose the results are given as follow.

Table 15: Distance of the faces of 200 Spanners measured to the nearest 0.1 mm

Distance(mm)	28.1-28.5	28.6-29.0	29.1-29.5	29.6-30.0	30.1-30.5	30.6-31.0	31.1-31.5	31.6-32.0
freq (f)	1	6	14	22	34	54	56	13

Estimate by calculation the median of this distribution

Soln: Since the intervals don't overlap

∴ Step (a): 28.6 - 28.5 = 0.1 . Step (b): $\frac{0.1}{2} = 0.05$

Step (c): $28.10 - 0.05 = 28.05$ and

28.5 + 0.05 = 28.55
29.0 + 0.05 = 29.05
29.5 + 0.05 = 29.55
30.0 + 0.05 = 30.05
30.5 + 0.05 = 30.55
31.0 + 0.05 = 31.05
31.5 + 0.05 = 31.55
32.0 + 0.05 = 32.05

Hence, we've,

Table 16: Computed Class Boundaries for Table 15

Dist.	Class-boundaries	frequency (f)
28.1 - 28.5	28.05 - 28.55	1
28.6 - 29.0	28.55 - 29.05	6
29.1 - 29.5	29.05 - 29.55	14
29.5 - 30.0	29.55 - 30.05	22
30.1 - 30.5	30.05 - 30.55	34
30.6 - 31.0	30.55 - 31.05	54
31.0 - 31.5	31.05 - 31.55	56
31.6 - 32.0	31.55 - 32.05	13
		$\Sigma f = 200$

class intervals do not overlap

Recall that for grouped data, the median $= \left[\frac{1}{2}(\Sigma f + 1)\right]^{th}$ number $= \left[\frac{1}{2}(200+1)\right]^{th} = 100\frac{1}{2}^{th}$ number $\left[\frac{1}{2}(N+1)\right]^{th}$

Counting along the f-lone from the top. Here

Using method 1: $L_1 = 30.55$, $f_{median} = 54$, $C = 31.05 - 30.55$

$(\Sigma f)_2 = 1 + 6 + 14 + 22 + 34 = 77$ and $N = \Sigma f = 200$

$$Median = L_1 + \left(\frac{N/2 - (\Sigma f)_2}{f_{median}}\right)C = 30.55 + \left(\frac{200/2 - 77}{54}\right)0.5$$

$= 30.762962\cancel{96} \Rightarrow 30.76$ (2 d.p.)

Using method 2: Median $= L_1 + \left(\frac{P_1 C}{f_{median}}\right)$

$L_1 = 30.55$, $P_1 = 100\frac{1}{2} - 77$

$= 23\frac{1}{2}$, $C = 0.5$, $f_{median} = 54$

$$Median = 30.55 + \left(\frac{23\frac{1}{2} \times 0.5}{54}\right) = 30.7695925 9$$

$\Rightarrow 30.77$ (2 d.p.)

Ex 14: Suppose the following are the age groups of 125 farmers on a Community. Construct the cummulative frequency curve of the distribution.

Table 1.7: The distribution of the age of 125 people

Age group	15-24	25-34	35-44	45-54	55-64
Freq (f)	10	25	45	30	15

Soln: Table 1.8: Computed class boundaries for Table 1.7

Age group	Class boundaries	f	Cummulative freq
15 - 24	14.5 - 24.5	10	10
25 - 34	24.5 - 34.5	25	35
35 - 44	34.5 - 44.5	45	80
45 - 54	44.5 - 54.5	30	110
55 - 64	54.5 - 64.5	15	125.
		$\Sigma f = 125$	

1 cm rep. 10 units on both x & y axis

Remarks: 1

(1) Percentage cummulative freq. can also be marked on the vertical axis & respective coordinate values on the horiz-axis read off from of curve - these read-off values are called the percentiles and are known as measure of partition.

(2) Percentage cummulative freq. are calc. as follows:

100% Percentile (P_{100}) = $\frac{100\%}{100} \times 125 = 125$ cummulative freq value

80% Percentile (P_{80}) = $\frac{80}{100} \times 125 = 100$ cummulative freq. value

75% Percentile (P_{75}) = $\frac{75}{100} \times 125 = 93.75$ cummulative freq value

50% percentile (P_{50}) = $\frac{50}{100} \times 125 = 62.5$ cummulative freq. value

The coordinate value of (P_{50}) marked, read off from the x-axis gives the median

20% percentile (P_{20}) = $\frac{20}{100} \times 125 = 25$ cummulative freq. values

0% percentile (P_0) = $\frac{0}{100} \times 125 = 0$

All these are indicated on the y-axis of figure 5 and their corresponding marks on the x-axis read off as follows

Lecture 16/03/23

Remark on relationship between Mean, median and ~~Empercen~~ mode

Emperical formula: Mean - mode = 3 (mean - median

Measure of ~~distribution~~ Partition

(a) Quartiles: This is a measure of partition which divides a distribution into equal parts: Q_1, Q_2 and Q_3. The Q_1 and Q_2, are called the lower and upper end quartiles respectively. Note ~~that~~ also that Q_2 is the same as the median. The measure can also be computed for both ungrouped and grouped data.

For ungrouped data

$Q_1 = \left(\frac{1}{4}(N+1)\right)$ The member of the distribution

$Q_2 = \left(\frac{2}{4}(N+1)\right)$ The member of the distribution

$Q_3 = \left(\frac{3}{4}(N+1)\right)$ the member of the distribution

for grouped data: use the formular

$$Q_i = LQ_i + \frac{(PQ_i C}{f_{Q_i}}$$

Where, i is the quartile required for LQ_i is the lower class boundary of the class of the quartile (i.e quartile class)

PQ_i is the position of the quartile in the quartile class, and f_{Qi} is the frequency of quartile class.

Remarks: If graphical solution is desired for the quartile estimates, the cummulative frequency curve (ogive) is used.

(b) Deciles and Percentiles: These are measures of partition which divides a distribution into 10 and 100 equal parts respectively.

If calculation method is required for a grouped frequency distribution.

f_{Qi} in the quartile formular is replaced by f_{Di} ~~or~~ f_{Pi};

PQ_i in the quartile formula is replaced by P_{Di} or P_{Pi}, while

LQ_i in the quartile formular is replaced by L_{Di} and L_{Pi} depending on whether it is decile and percentile, one is interested in calculating. Hence we have;

$$D_i = L_{Di} + \left(\frac{P_{Di} C}{f_{Di}}\right) \quad i = 1, 2, 3 \ldots 9$$

Where;

i is the decile being referred to

L_{Di} is the lower class boundary of the class of the decile (i.e decile class)

P_{Di} is the position of the decile in the decile class and

f_{Di} is the frequency of the decile class.

In the same light

$$P_i = L_{Pi} + \left(\frac{P_{Pi} C}{f_{Pi}}\right) \quad i = 1, 2, 3 \ldots 9$$

Where i is the percentile referred to

L_{Pi} is the lower class boundary of the class of the percentile (i.e percentile class)

P_{Pi} is the position of the percentile in the percentile class, and

f_{Pi} is the frequency in the percentile class.

Example 15: Find by (a) Calculation (b) graphical method the, (i) Q_2 (ii) D_1 and (iii) P_{30} of the distribution of table 19

Table 19: Grouped frequency distribution of 50 items

Profit ₦	1-10	11-20	21-30	31-40	41-50	51-60
frequency (f)	6	6	12	11	0	50

$\Sigma f = 50$

Solution

By calculation we have table 20 for grouped data

Table 20: Grouped data frequency distribution of 50 items

Profit (₦)	Class boundaries	Frequency	Cummulative frequency
1-10	0.5 - 10.5	6 $\rightarrow D_1$	6
11-20	10.5 - 20.5	6	12
21-30	20.5 - 30.5	12 $\rightarrow P_{30}$	24
31-40	30.5 - 40.5	11 $\rightarrow Q_2$	35
41-50	40.5 - 50.5	10	45
51-60	50.5 - 60.5	5	50

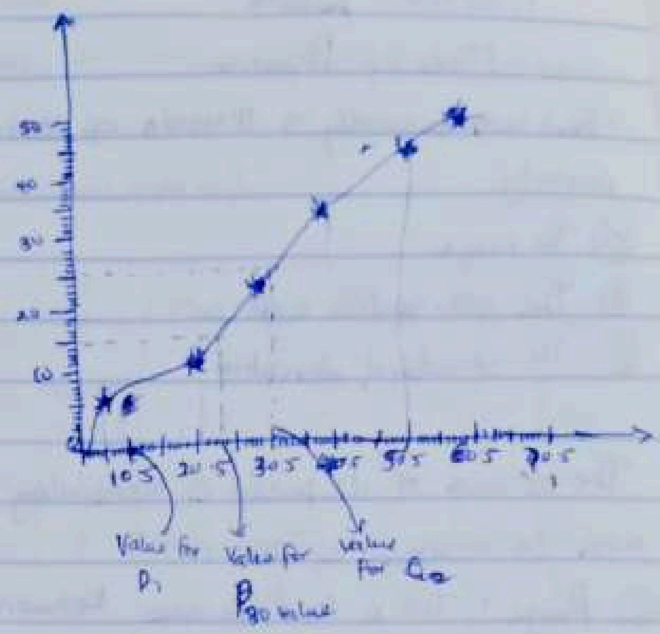
(i) $Q_2 = \left(\frac{2}{4}N\right) = \left(\frac{2}{4}\times 50\right) = 25$th number of distribution

(ii) $D_1 = \left(\frac{1}{10}N\right) = \frac{1}{10}\times 50 = 5$th number of distribution and

(iii) $P_{30} = \left(\frac{30}{100}N\right) = \frac{30}{100}\times 50 = 15$th number of the distribution.

Class boundary	0.5 - 10.5	10.5 - 20.5	20.5 - 30.5	30.5 - 40.5	40.5 - 50.5	50.5 - 60.5
Cum. frequency	6	12	24	35	45	50

Scale: 1cm represents 10 units on both x and y axis respectively



$Q_2 = L_{Q_2} + \frac{P_{Q_2} C}{f_{Q_2}}$

$= 30.5 + \frac{(25-24)}{11}\times 10 =$

$D_1 = L_{D_1} + \frac{P_{D_1} C}{f_{D_1}} = 0.5 + \frac{5\times 10}{6} =$

$P_{30} = L_{P_{30}} + \frac{P_{P_{30}}}{f_{P_{30}}} = 20.5 + \left(\frac{15-12}{12}\times 10\right) =$

17/05/23 Lecture

Mode of Dispersion

There are basically 3 measure of dispersion namely:

(a) The range

(b) The interquatile range and

(c) The standard deviation.

The measure of dispersion is taken along side with the measure of position

(a) Range: This is the difference between the highest and lowest values of a variable. e.g for marks 35, 36, 46, 48, 58, 63, 65, 68, 75 and 84, the range is $84 - 35 = 49$

It depends only on two extreme values of a data set

(b) Interquartile Range: This is the difference between the upper and lower quartile i.e $Q_3 - Q_1$.

Sometimes the semi interquartile range is used instead and this is given by $\frac{1}{2}(Q_3 - Q_1)$. It is not related mathematically to the mean

(c) Standard Deviation (SD) of an Ungrouped Frequency

Standard Deviation: The standard deviation is the most efficient measure of dispersion and the most widely used because it takes every value of the variable into account and is related numerically to the mean. It is defined as follows:

(i) For a set of numbers:

$$SD = s = \sqrt{\frac{\Sigma d^2}{n}} = \sqrt{\frac{\Sigma(x-\bar{x})^2}{n}} \quad \cdots (8)$$

where $d = x - \bar{x}$ while

(ii) For a frequency distribution as

$$S.D = S = \sqrt{\frac{\Sigma fd^2}{n} - \left(\frac{\Sigma fd}{n}\right)^2} \quad \cdots\cdots (9)$$

Equation 9 is the shortest method for calculating standard deviation. The idea of an assumed mean is also used, such that one will be taking the deviations from the assumed mean rather than taking the deviation from the mean.

Example 16: Calculate the mean and S.D of the distribution table 21

Table 21: Ungrouped frequency distribution of 70 items

x	15	25	35	45	55	65	75	
f	1	4	12	24	18	8	3	$\Sigma f = 70$

Solution

By calculation, we have Table 22 as follows:

Table 22: Computation table for the estimation of S.D by the short method.

x	f	$d = x - A$	fd	d^2	fd^2
15	1	-30	-30	900	900
25	4	-20	-80	400	1600
35	12	-10	-120	100	1200
45 A	24	0	0	0	0
55	18	10	180	100	1800
65	8	20	160	400	3200
75	3	30	90	900	2700
	$\Sigma f = 70$		$\Sigma fd = 200$		$\Sigma fd^2 = 11,400$

It's best to choose the assumed mean (i.e A) to be the middle x value

The mean $= \bar{x} = A + \frac{\Sigma fd}{n}$

$= 45 + \frac{200}{70} = 47.85714286$

$\simeq 47.9$ (1.dp)

$S.D = S = \sqrt{\frac{\Sigma fd^2}{n} - \left(\frac{\Sigma fd}{n}\right)^2}$

$\Rightarrow \sqrt{\frac{11400}{70} - \left(\frac{200}{70}\right)^2}$

$\Rightarrow 12.43759935$

$\Rightarrow 12.44$ (2d.p)

Remarks:

(a) The square of the standard deviation, denoted by v^2, is called variance

(b) Always round up the standard deviation value to 2 more decimal places to the data

(c) For data sets that has unit, the estimate of the mean and standard deviation will both have the same unit as the data.

(d) Standard Deviation (SD) of a Grouped Frequency

The same method as that of ungrouped frequency distribution is used but in this case, the deviation, are taken from the class Centres

Example 17: Find the mean and SD of the distribution on Table 23.

Table 23: Grouped frequency distribution of Height (in cm) of 110 people.

Height (cm)	150–154	155–159	160–164	165–169	170–174	175–179	180–184
Frequency (f)	4	14	26	32	21	10	3

$\Sigma f = 110$

Solution:

Table. 24 Computation Table for the estimation of SD from table 23 by short method (i.e the assume mean approach).

Class limits	class centres	f	$d = x - A$	d^2	fd	fd^2
150–154	$\frac{150+154}{2} = 152$	4	$152 - 167 = -15$	225	-60	900
155–159	157	14	10	100	-140	1400
160–164	162	26	5	25	-130	650
165–169	167	32	0	0	0	0
170–174	172	21	5	25	105	525
175–179	177	10	10	100	100	100
180–184	182	3	15	225	45	675
		$\Sigma f = 110$			$\Sigma fd = -80$	$\Sigma fd^2 = 5150$

The Mean $= A + \frac{\Sigma fd}{n}$

$\Rightarrow 167 - \left(\frac{80}{110}\right) = 166.2727273$

$\simeq 166.3$cm (1 dp)

and

$SD = S = \sqrt{\frac{\Sigma fd^2}{n} - \left(\frac{\Sigma fd}{n}\right)^2}$

$= \sqrt{\frac{5150}{110} - \left(\frac{-80}{110}\right)^2}$

$= \sqrt{\frac{5150}{110} - \frac{6400}{12100}}$

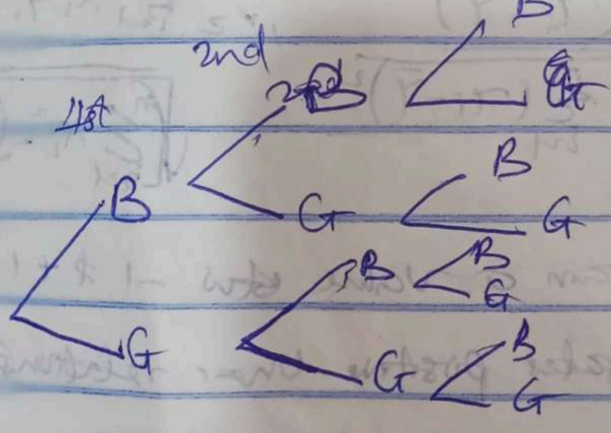
$= \sqrt{\frac{560100}{12100}} = 6.803620813$

$\simeq 6.80$cm (2 dp

Probability

S or $\Omega = \{1, 2, 3, 4, 5, 6\}$ — prob of rolling a dice

probability of an event happening is 2^n



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Classical probability, $P(A) = \frac{n_A}{n}$

Ex: A = rolling (tossing) of a dice, $n = 6$, $n_A = 1$

$P(A) = 1/6$

$\frac{n_A}{n} \leq 1$

$Pr(A \text{ and } B) = Pr(A) + Pr(B) - Pr(A \cap B)$

$Pr(A \text{ or } B) = Pr(A) + Pr(B)$

Addition rule

$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$

$P(A \text{ or } B \text{ or } C) = P(A) + P(B) + P(C) - [P(A \& B) + P(A \& C) + P(B \& C)] + P(A \& B \& C)$

Mutually exclusive - can't occur simultaneously

$P(A \overset{\text{and } C}{\cap} B) = 0$ $P(A \& B \& C \& D)$

$P(A \text{ or } B) = P(A) + P(B)$

$P(A \text{ or } B \text{ or } C) = P(A) + P(B) + P(C)$

Exhaustive events

If events A & B are mutually exclusive & collectively exhaustive. $P(A) + P(B) = 1$

Independent events

$P(A \& B) = P(A) \times P(B)$

Conditional probability is of an event B in relationship to event A is d probability that event B occurs after event A has already occured.

$$P(B/A) = \frac{P(A \cap B)}{P(A)} ; P(A) \neq 0$$

If probability of B given A ($P(B/A) \neq P(B)$, they're dependent.

$P(B/A) = P(B)$ it's dependent.