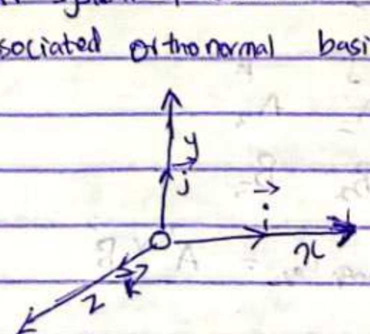
3rd April, 2024

VECTOR CALCULUS

Let $O(x, y, z)$ denote the mass of a rectilinear system of coordinates with the associated orthonormal basis $\vec{i}, \vec{j}, \vec{k}$



(I) $\vec{i}\cdot\vec{i} = 1 = \vec{j}\cdot\vec{j} = \vec{k}\cdot\vec{k} = 1$

(II) $\vec{i}\cdot\vec{j} = 0 = \vec{j}\cdot\vec{k} = \vec{k}\cdot\vec{j} = 0$

(III) $\vec{i}\times\vec{j} = \vec{k}$, $\vec{j}\times\vec{i} = -\vec{k}$, $\vec{j}\times\vec{k} = \vec{i}$,
$\vec{k}\times\vec{j} = -\vec{i}$, $\vec{k}\times\vec{i} = \vec{j}$, $\vec{i}\times\vec{k} = -\vec{j}$
$\vec{i}\times\vec{i} = 0 = \vec{j}\times\vec{j} = \vec{k}\times\vec{k} = 0$

(IV) Kronecker Delta (δ_{mn})

$$\delta_{mn} = \vec{e}_m\cdot\vec{e}_n = \begin{cases} 1, & \text{if } \vec{e}_m = \vec{e}_n \\ 0, & \text{if } \vec{e}_m \neq \vec{e}_n \end{cases}$$

Where $\vec{e}_m, \vec{e}_n = i, j, k$

(V) Levi - Civital Symbol

$$\varepsilon_{ijk} = (\vec{e}_m \times \vec{e}_n)\cdot\vec{e}_q = \begin{cases} 1, & \text{if } \vec{e}_m \neq \vec{e}_n \neq \vec{e}_q;\ p(\vec{e}_m\,\vec{e}_n\,\vec{e}_q) \text{ is even} \\ 0, & \text{if any of } \vec{e}_m, \vec{e}_n, \vec{e}_q \text{ equal} \\ -1, & \text{if } \vec{e}_m \neq \vec{e}_n \neq \vec{e}_q;\ p(\vec{e}_m\,\vec{e}_n\,\vec{e}_q) \text{ is odd} \end{cases}$$

Permutation

Where $P(\vec{e}_m\,\vec{e}_n\,\vec{e}_q)$ is cyclic Permutation of $\vec{e}_m, \vec{e}_n, \vec{e}_q$ and $\vec{e}_m, \vec{e}_n, \vec{e}_q = \vec{i}, \vec{j}, \vec{k}$

Meaning:

$(\vec{i}\times\vec{j})\cdot\vec{k} = \vec{k}\cdot\vec{k} = 1$ (even permutation)

$(\vec{j}\times\vec{i})\cdot\vec{k} = -\vec{k}\cdot\vec{k} = -1$ (odd)

$(\vec{i}\times\vec{j})\cdot\vec{i} = \vec{k}\cdot\vec{i} = 0$

When moving $\circlearrowright$ $i\ j\ k$ (even)

When moving $\circlearrowleft$ $i\ j\ k$ (odd)

$\therefore$ $\vec{i},\vec{j},\vec{k}$, $\vec{j},\vec{k},\vec{i}$, $\vec{k},\vec{i},\vec{j}$ even permutation
$\vec{i}\times\vec{k},\vec{j}$, $\vec{k},\vec{j},\vec{i}$, $\vec{j},\vec{i},\vec{k}$ odd permutation

(VI) $\varepsilon_{ijk} = (\vec{e}_i\times\vec{e}_j)\cdot\vec{e}_k \Rightarrow \vec{e}_i\times\vec{e}_j = \varepsilon_{ijk}\vec{e}_k$

(VII) Note that repeated index denotes Summation

(a) $a_m b_m = a_1b_1 + a_2b_2 + a_3b_3 \Rightarrow \sum_{m=1}^{3} a_m b_m$

(b) $A_{ik}B_{kj} = A_{i1}B_{1j} + A_{i2}B_{2j} + A_{i3}B_{3j} = \sum_{k=1}^{3} A_{ik}B_{kj}$

(c) ~~$A_{mn}B_{np}C_p = \sum_{n=1}^{3}\left(\sum_{p=1}^{3}(A_{mp}B_{np})C_p\right)$~~

~~wrt $n =$~~

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(C) $A_{mn}B_{np}C_p = \sum_{n=1}^{3}\sum_{p=1}^{3} A_{mn}B_{np}C_p$

wrt $n =$ $A_{m1}B_{1p}C_p + A_{m2}B_{2p}C_p + A_{m3}B_{3p}C_p$

wrt $p =$ $A_{m1}B_{11}C_1 + A_{m1}B_{12}C_2 + A_{m1}B_{13}C_3$
$+ A_{m2}B_{21}C_1 + A_{m2}B_{22}C_2 + A_{m2}B_{23}C_3$
$+ A_{m3}B_{31}C_1 + A_{m3}B_{32}C_2 + A_{m3}B_{33}C_3$

Note that: $1 \to \vec{i}$, $2 \to \vec{j}$, $3 \to \vec{k}$

$\varepsilon_{11} = \vec{i}\times\vec{i} = 0$ $\varepsilon_{12} = \vec{i}\times\vec{j} = k$, $\varepsilon_{12} = 0$, $\varepsilon_{123} = (\vec{i}\times\vec{j})\cdot k = 1$ etc

$\delta_{11} = \vec{i}\cdot\vec{i} = 1$, $\delta_{12} = \vec{i}\cdot\vec{j} = 0 \ldots$ etc

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(VIII) $\vec{A}\cdot\vec{B} = A_m \vec{e}_m \cdot B_n \vec{e}_n$

$\Rightarrow A_m B_n \vec{e}_m \cdot \vec{e}_n$

$\Rightarrow A_m B_n \delta_{mn}$

$\Rightarrow A_1 B_n \delta_{1n} + A_2 B_n \delta_{2n} + A_3 B_n \delta_{3n}$

$\Rightarrow A_1 B_1 \delta_{11} + A_1 B_2 \delta_{12} + A_3 B_3 \delta_{13}$
$+ A_2 B_1 \delta_{21} + A_2 B_2 \delta_{22} + A_2 B_3 \delta_{23}$
$+ A_3 B_1 \delta_{31} + A_3 B_2 \delta_{32} + A_3 B_3 \delta_{33}$

$\Rightarrow A_1 B_1 \delta_{11} + 0 + 0 + 0 + A_2 B_2 \delta_{22} + 0$
$+ 0 + 0 + A_3 B_3 \delta_{33}$

$\Rightarrow A_1 B_1 + A_2 B_2 + A_3 B_3$

OR

Note: $\vec{A}\cdot\vec{B} = A_m B_n \delta_{mn}$
(if $m = n = m$)

Then $\Rightarrow A_m B_m \delta_{mm}$ ($\delta_{mm} = 1$)

$\Rightarrow A_m B_m = A_1 B_1 + A_2 B_2 + A_3 B_3$

Note: $\varepsilon_{mnq} = (\vec{e}_m \times \vec{e}_n) \cdot \vec{e}_q$

$\Rightarrow \vec{e}_m \times \vec{e}_n = \varepsilon_{mnq} \vec{e}_q$

(IX) $\vec{A}\times\vec{B} = A_m \vec{e}_m \times B_n \vec{e}_n$

$= A_m B_n \vec{e}_m \times \vec{e}_n$

$= A_m B_n \varepsilon_{mnq} \vec{e}_q$

$= A_1 B_n \varepsilon_{1nq} \vec{i}_q + A_2 B_n \varepsilon_{2nq} \vec{i}_q + A_3 B_n \varepsilon_{3nq} \vec{i}_q$

$= A_1 B_2 \varepsilon_{12q} \vec{i}_q + A_1 B_3 \varepsilon_{13q} \vec{i}_q + A_2 B_1 \varepsilon_{21q} \vec{i}_q$
$+ A_2 B_3 \varepsilon_{23q} \vec{i}_q + A_3 B_1 \varepsilon_{31q} \vec{i}_q + A_3 B_2 \varepsilon_{32q} \vec{i}_q$

$e_3 = k,\ e_2 = j,\ e_1 = i$

$= A_1 B_2 \varepsilon_{123} \vec{k} + A_1 B_3 \varepsilon_{132} \vec{j} + A_2 B_1 \varepsilon_{213} \vec{i}$
$+ A_2 B_{231} i + A_3 B_1 \varepsilon_{312} \vec{j} + A_3 B_2 \varepsilon_{321} \vec{i}$

$= (A_2 B_3 - A_3 A_2) i - (A_1 B_3 - A_3 B_1) j$
$+ (A_1 B_2 - A_2 B_1) \vec{k}$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix}$$

$= A_m B_n \varepsilon_{mnq} \vec{i}_q = \vec{A} \times \vec{B}$

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Scalar Field

Physical quantity that we characterize by magnitude only are referred to as scalars. Example: temperature, mass, speed etc.

A scalar field is a region of space in which every point of the region is assigned a scalar.

Thus, a scalar field is defined by a single-valued scalar function, say $\phi = \phi(x, y, z)$, where x, y, z is the coordinates of arbitrary of the region.

The Gradient Vector

Let P and Q be neighbouring points in a scalar field, defined by a single value scalar function ϕ. If $\phi(P)$ and $\phi(Q)$ denote the value of scalar function ϕ at P and Q respectively.

The gradient of ϕ at P, denoted

$\nabla\phi$, is defined by:

$$\nabla\phi = \lim_{Q \to P} \frac{\phi(Q) - \phi(P)}{PQ}$$

provided the limit exists.

In rectilinear Coordinates system the above definition assumes the form

$$\boxed{\nabla\phi := \vec{i}_m \frac{\partial\phi}{\partial x_m} = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}}$$

Example: If $\phi = 8x^2yz^3$, find $\Delta\phi$.

Solution

$$\phi = 8x^2yz^3$$

$$\nabla\phi = \vec{i}_m\frac{\partial\phi}{\partial x_m} = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$

$$\nabla\phi = 16xyz^3\vec{i} + 8x^2z^3\vec{j} + 24x^2yz^2\vec{k}$$

Example 2: If $\phi = 2xy^2z^3$ and $\psi = x\cos z$. Find $\nabla(\nabla\phi \cdot \nabla\psi)$

Solution

$$\nabla\phi = \vec{i}_m\frac{\partial\phi}{\partial x_m} = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$

$$= 2y^2z^3\vec{i} + 4xyz^3\vec{j} + 6xy^2z^2\vec{k}$$

$$\nabla\psi = \vec{i}_m\frac{\partial\psi}{\partial x_m} = \vec{i}\frac{\partial\psi}{\partial x} + \vec{j}\frac{\partial\psi}{\partial y} + \vec{k}\frac{\partial\psi}{\partial z}$$

$$= \cos z\,\vec{i} - x\sin z\,\vec{k}$$

$$\nabla\phi \cdot \Delta\psi = 2y^2z^3\cos z - 6x^2y^2z^2\sin z$$

$$\nabla(\nabla\phi \cdot \nabla\psi) = \vec{i}_m\frac{\partial}{\partial x_m}(\nabla\phi \cdot \nabla\psi)$$

$$= \vec{i}\frac{\partial}{\partial x}(\nabla\phi \cdot \nabla\psi) + \vec{j}\frac{\partial}{\partial y}(\nabla\phi \cdot \nabla\psi) + \vec{k}\frac{\partial}{\partial z}(\nabla\phi \cdot \nabla\psi)$$

$$= -12xy^2z^2\sin z\,\vec{i} + 4yz^2(z\cos z - 3x^2\sin z)\vec{j} + \left(2y^2[3z^2\cos z - z^3\sin z] - 6x^2y^2[2z\sin z + z^2\cos z]\right)\vec{k}$$

When the scalar field is constant at every surface, it is called a levelled surface. But in physics is called equipotential surface.

In the above examples, it is straight forward to infer that the del-operator ∇ in rectangular coordinates system, is given by:

$$\nabla = \vec{i}_m\frac{\partial}{\partial x_m} = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}$$

Example 3: show that the gradient vector $\nabla\phi$ is ~~proportional~~ perpendicular to the surface ϕ $\phi = \phi(x,y,z) = C$, where C is a constant

Solution: $\phi = \phi(x,y,z) = C$

$$d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz = 0$$

$$= \frac{\partial\phi}{\partial x_n}dx_n\delta_{mn} = \vec{i}_m\frac{\partial\phi}{\partial x_m}\cdot\vec{i}_n dx_n = 0$$

$$= \nabla\phi \cdot d\vec{r} = 0$$

Since $d\vec{r} \neq \vec{0}$ and $d\vec{r}$ is an elemental vector on surface $\phi = \phi(x,y,z) = C$, therefore, the gradient vector $\nabla\phi$ is perpendicular to surface ~~ϕ = constant~~

$$\phi = \phi(x,y,z) = \text{constant}$$

"When you have a function that describes a level surface, the perpendicular vector of the surface is the gradient of that surface."

Example 4:

Find a unit normal to the surface $2x^2 + 4yz - 5z^2 = -10$ at point $P(3, -1, 2)$

Solution:

A vector perpendicular to the surface $\phi =$ constant is the gradien vector $\nabla\phi$ ~~at point p(3, -1, 2)~~

$$\phi = 2x^2 + 4yz - 5z^2 = -10$$

$$\nabla\phi = \vec{i}_m \frac{\partial\phi}{\partial x_m} = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$

$$\nabla\phi = 4x\vec{i} + 4z\vec{j} + (4y - 10z)\vec{k}$$

$$\nabla\phi\Big|_{(3,-1,2)} = 12\vec{i} + 8\vec{j} - 24\vec{k}$$

The unit ~~vector~~ normal to the surface

$$\frac{\nabla\phi}{|\nabla\phi|} = \frac{(12\vec{i} + 8\vec{j} - 24\vec{k})}{\sqrt{12^2 + 8^2 + (-24)^2}}$$

$$= \frac{12\vec{i} + 8\vec{j} - 24\vec{k}}{28} = \frac{3}{7}\vec{i} + \frac{2}{7}\vec{j} - \frac{6}{7}\vec{k}$$

Directional Derivative

The directional derivative of a scalar function ϕ at point $\vec{x}$ in the direction of unit vector $\hat{u}$, denoted by $\nabla u\phi$, is defined by:

$$\nabla u\phi = \lim_{h\to 0} \frac{\phi(\vec{x} + h\hat{u}) - \phi(\vec{x})}{h}$$

$$= \nabla\phi \cdot \hat{u} = \frac{\nabla\phi \cdot \vec{u}}{|\vec{u}|}$$

Divergence of Vector

Let Σ be a closed surface, enclosing the volume Ω, the divergence of $\vec{A}$, denoted $\nabla \cdot \vec{A}$ is defined by

$$\nabla \cdot A = \lim_{\Omega\to 0} \frac{\left(\oiint \vec{A} \cdot |\vec{\Sigma}|\right)}{\Omega}$$

Provided that the limit exist.

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In rectilinear coordinates system, the above definition assumes the form

$$\nabla\cdot\vec{A} = \vec{i}_m \frac{\partial}{\partial x_m}\cdot A_n\vec{i}_n = \frac{\partial A_n}{\partial x_m}\vec{e}_m\cdot\vec{e}_n$$

$$= \frac{\partial A_n}{\partial x_m}\cdot\delta_{mn} = \frac{\partial A_m}{\partial x_m}$$

$$\Rightarrow \frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}$$

Example 5: If $\vec{A} = 3x^2yz^3\vec{i} - 4xy^3z^2\vec{j} + e^{xyz}\vec{k}$, find the $\nabla\cdot\vec{A}$ (divergence of A)

Solution

$$\nabla\cdot\vec{A} = \vec{i}_m\frac{\partial}{\partial x_m}\cdot A_n\vec{i}_n = \frac{\partial A_m}{\partial x_m} = \frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}$$

$$= \frac{\partial}{\partial x}(3x^2yz^3) + \frac{\partial}{\partial y}(-4xy^3z^2) + \frac{\partial}{\partial z}(e^{xyz})$$

$$= 6xyz^3 - 12xy^2z^2 + xye^{xyz}$$

Example 6: If $\vec{F} = (3x^2y + z^3)\vec{i} - (xz^3 + y^4)\vec{j}$, find $\nabla(\nabla\cdot\vec{F})$

Solution

$$\nabla\cdot\vec{F} = \vec{i}_m\frac{\partial}{\partial x_m}\cdot f_n\vec{i}_n = \frac{\partial f_m}{\partial x_m}$$

$$= \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

$$= \frac{\partial}{\partial x}(3x^2y + z^3) - \frac{\partial}{\partial y}(xz^3 + y^4) + \frac{\partial}{\partial z}(0)$$

$$\nabla\cdot\vec{F} = 6xy - 4y^3$$

$$\nabla(\nabla\cdot\vec{F}) = \vec{i}_m\frac{\partial}{\partial x_m}(\nabla\cdot\vec{F})$$

$$= \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)(6xy - 4y^3)$$

$$= 6y\vec{i} + (6x - 12y^2)\vec{j}$$

Note: A single-valued, differentiable vector function $\vec{F}$ is said to be SOLENOIDAL if

$$\nabla\cdot\vec{F} = 0$$

Example 7: Show that the vector $\vec{F} = 3y^2z\vec{i} - 8zx^2\vec{j} + \sin x\vec{k}$ is solenoidal

Solution

$$\nabla\cdot\vec{F} = \vec{i}_m\frac{\partial}{\partial x_m}\cdot f_n\vec{i}_n = \frac{\partial f_m}{\partial x_m}$$

$$= \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

$$= \frac{\partial}{\partial x}(3y^2z) + \frac{\partial}{\partial y}(-8zx^2) + \frac{\partial}{\partial z}(\sin x)$$

$$= 0 + 0 + 0 = 0$$

Therefore, $\vec{F}$ is solenoidal

Rotor / Curl of Vector Functions

Let Σ be a closed surface, enclosing the volume Ω, the rotor of vector $\vec{F}$, denoted by $\nabla\times\vec{F}$, is defined by

$$\nabla\times\vec{F} = \lim_{\Omega\to 0}\left(\frac{\oint\!\!\oint_{\Sigma} d\vec{\Sigma}\times\vec{F}}{\Omega}\right)$$

In rectangular coordinates system, the above definition is given by

$$\nabla\times\vec{F} = \vec{i}_m\frac{\partial}{\partial x_m}\times f_n\vec{i}_n = \frac{\partial f_n}{\partial x_m}\vec{e}_m\times\vec{e}_n$$

$$= \frac{\partial F_n}{\partial x_m}\varepsilon_{mnq}\vec{e}_q$$

$$= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\vec{i} - \left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right)\vec{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\vec{k}$$

Note: If a single-valued, differentiable vector function $\vec{F}$ is IRROTATIONAL, then
$$\nabla \times \vec{F} = \vec{0}$$

Example 8: Given that $\vec{H} = 3y^4\vec{i} - \cos z\vec{j} + e^{3x^2}\vec{k}$
Find the $\nabla \times \vec{H}$

Solution

$$\nabla \times \vec{H} = \vec{i}_m \frac{\partial}{\partial x_m} \times H_n \vec{i}_n = \frac{\partial H_n}{\partial x_m}\varepsilon_{mnq}\vec{i}_q$$

$$= \left(\frac{\partial}{\partial y}H_3 - \frac{\partial H_2}{\partial z}\right)\vec{i} - \left(\frac{\partial H_3}{\partial x} - \frac{\partial H_1}{\partial z}\right)\vec{j}$$
$$+ \left(\frac{\partial H_2}{\partial x} - \frac{\partial H_1}{\partial y}\right)\vec{k}$$

$$\nabla \times \vec{H} = (0 - \sin z)\vec{i} - (6xe^{3x^2} - 0) + (0 - 12y^3)\vec{k}$$
$$= -(\sin z)\vec{i} - (6xe^{3x^2})\vec{j} - 12y^3\vec{k}$$

Standard Results

Suppose ϕ and $\vec{F}$ are differentiable scalar and vector functions respectively.

(i) $\nabla \cdot (\nabla \times \vec{F}) = 0$

(ii) $\nabla \times (\nabla \phi) = \vec{0}$

The above results in (i) and (ii) are summarized by DOROGO
(is zero, is zero; Divergence of Rotor of Gradient)

Formulae

Suppose ϕ is a differentiable scalar function and let $\vec{A}$ and $\vec{B}$ be differentiable vector functions. Then,

(i) $\nabla \cdot (\phi\vec{A}) = \phi\nabla \cdot \vec{A} + \vec{A} \cdot \nabla\phi$

(ii) $\nabla \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\nabla \times \vec{A}) - \vec{A} \cdot (\nabla \times \vec{B})$

(iii) $\nabla \times (\phi\vec{A}) = \phi(\nabla \times \vec{A}) + (\nabla\phi \times \vec{A})$

(iv) $\nabla \times (\vec{A} \times \vec{B}) = (\nabla \cdot \vec{B})\vec{A} - (\nabla \cdot \vec{A})\vec{B} + (\vec{B} \cdot \nabla)\vec{A} - (\vec{A} \cdot \nabla)\vec{B}$

(v) $\nabla(\vec{A} \cdot \vec{B}) = \vec{A} \times (\nabla \times \vec{B}) + B \times (\nabla \times \vec{A}) + (\vec{A} \cdot \nabla)\vec{B} + (\vec{B} \cdot \nabla)\vec{A}$

Example 9: Suppose $\vec{\omega}$ is a constant vector and let vector function $\vec{H}$ be irrotational. Show that
$\vec{G} = \vec{\omega} \times \vec{H}$ is solenoidal

Solution

$$\nabla \cdot \vec{G} = \nabla \cdot (\vec{\omega} \times \vec{H})$$
$$= \vec{H} \cdot (\nabla \times \vec{\omega}) - \vec{\omega} \cdot (\nabla \times \vec{H})$$
$$= 0 - 0 = 0$$

$\vec{G}$ is solenoidal

Lapacian of Scalar Function

Let ϕ be twice-differentiable scalar function. The lapacian of ϕ, denoted $\nabla^2\phi$, is denoted by

$$\nabla^2\phi = \nabla \cdot (\nabla\phi) = \vec{i}_m \frac{\partial}{\partial x_m} \cdot \left(\vec{i}_n \frac{\partial \phi}{\partial x_n}\right)$$

$$= \frac{\partial^2 \phi}{\partial x_m \partial x_n} \cdot \vec{i}_m \cdot \vec{i}_n = \frac{\partial^2 \phi}{\partial x_m \partial x_n} \cdot \delta_{mn}$$

$$= \frac{\partial^2 \phi}{\partial x_m^2} = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}$$

Example 10: Given that $\phi = 3x^4 \cos y e^z$

Find $\nabla^2 \phi$

Solution

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x_m^2} = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}$$

$$= 36x^2 \cos y e^z - \cancel{3x^4 \cos y e^z} + \cancel{3x^4 \cos y e^z}$$

$$= \underline{36x^2 \cos y e^z}$$

Observation

Revision

The Ha

14th April, 2024

Observation

→ The base vectors i, j, k forms an ORTHOGONAL set

1. They are pair-wise ORTHOGONAL
2. $|i| = 1 = |j| = 1 = |k|$

③ ① The coordinate ~~plane~~ Line of the cartesian system are ~~is STRAIGHT LINES~~ STRAIGHT LINES.

* The base vectors ijk are tangent to the coordinate line.

② The base vectors are i, j, k have perspectively directions of the tangents to the x-axis, y-axis, z axis. These direction are as we ~~~~ know are also, perspectively in the (direction) of the +ve x-axis, y-axis, z-axis

Coordinate surfaces

Consider the cartesian system (x, y, z). Its coordinate surfaces are

$$\begin{cases} x = \text{constant} \\ y = \text{constant} \\ z = \text{constant} \end{cases}$$

The xy-plane ($z=0$), ~~yz-plane and~~ yz ($x=0$) plane and xz ($y=0$) plane are the coordinate surface for the cartesian system

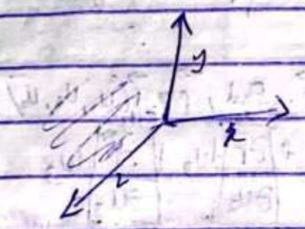
The coordinate variable are $u_1\,u_2\,u_3$

$f(x,y,z) = 0$ $\quad \nabla f \cdot$

→ Coordinate lines are the pair wise lines of intersection of coordinate surfaces. In the case of the cartesian system when has its coordinate surfaces as planes, the coordinate lines are clearly straight lines?

AN ALTERNATIVE VIEW OR BASE VECTORS

The normal of yz plane is $\cdot\, i(x)$ — Harmotonian of x



$$i = \nabla x$$
$$j = \nabla y$$
$$k = \nabla z$$

$$\nabla x = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z} \right) x = i\frac{\partial x}{\partial x} \Rightarrow i$$

→ The alternative view of base vectors is to consider each of them as normal vectors to coordinate surfaces. In the cartesian system, the i base vector is normal to the coordinate surface $(x=0)$, i.e $\nabla x = i$.
j is normal to the surface $(y=0)$ & $\nabla y = j$

$i = \{1,0,0\}$
$y = \{0,1,0\}$
$z = \{0,0,1\}$

Curvilinear systems of coordinate

→ We shall refer to the cartesian system as a global system of cordinate

→ Curvilinear system will sometimes be preffered to as local system of coordinate

→ Consider the curvilinear coordinate system (u_1, u_2, u_3)
The coordinate variable are u_1, u_2, u_3

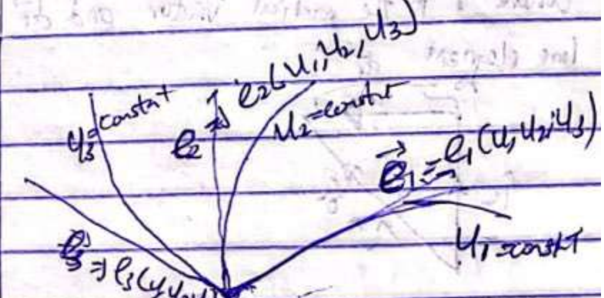


fig1: figure of a curvilinear system (u_1,u_2,u_3) are base vectors at a point P

The base vectors for curvilinear system are functions of their cordinate systems variables

15th April 2024

Curvilinear Coordinate Systems II

Consider a curvilinear coordinate system (u_1, u_2, u_3) such that referral to the coordinates x, y, z of the Cartesian or global system

$$(1)\quad \begin{cases} u_1 = u_1(x,y,z) \\ u_2 = u_2(x,y,z) \\ u_3 = u_3(x,y,z) \end{cases}$$

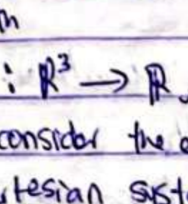
with

$$U_j : \mathbb{R}^3 \to \mathbb{R};\ j = 1,2,3$$

Now, consider the elemental arc length ds in the cartesian system:

$$d^2s = d\vec{r} \cdot d\vec{r}$$

Where $\vec{r}$ is the radial vector and $d\vec{r}$ is the line element



If $\vec{r} = x\underline{i} + y\underline{j} + z\underline{k}$, then the line element $d\vec{r}$ is

$$(2)\quad d\vec{r} = dx\underline{i} + dy\underline{j} + dz\underline{k}$$

SIDE COMMENTS

$$d\vec{r} = d(x\underline{i} + y\underline{j} + z\underline{k}) = d(x\underline{i}) + d(y\underline{j}) + d(z\underline{k})$$
$$= dx\underline{i} + x\cancel{d\underline{i}} + dy\underline{j} + y\cancel{d\underline{j}} + dz\underline{k} + z\cancel{d\underline{k}}$$

Therefore,

$$ds^2 = d\vec{r} \cdot d\vec{r} = dx^2 + dy^2 + dz^2$$

$$(dx, dy, dz)\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\boxed{\text{METRIC MATRIX}}}\begin{pmatrix} dx \\ dy \\ dz \end{pmatrix}$$

→ determines the curvature of a system

$$(3)\quad \text{Alt}\ (dx, dy, dz)\begin{pmatrix} i\cdot i & i\cdot j & i\cdot k \\ j\cdot i & j\cdot j & j\cdot k \\ k\cdot i & k\cdot j & k\cdot k \end{pmatrix}\begin{pmatrix} dx \\ dy \\ dz \end{pmatrix}$$

From eqn. 3, we observe that the metric matrix of the three dimensional Cartesian system is the 3×3 unit matrix. One can replicate the above concept for the three dimensional curvilinear system (u_1, u_2, u_3)

Following the same procedure, we note that

$$(ds)^2 = d\vec{r} \cdot d\vec{r}.\ \text{But, } \vec{r} = \vec{r}(u_1, u_2, u_3)$$

$$d\vec{r} = \underbrace{\boxed{\frac{\partial \vec{r}}{\partial u_1}}}_{\vec{e}_{u_1}} du_1 + \underbrace{\boxed{\frac{\partial \vec{r}}{\partial u_2}}}_{\vec{e}_{u_2}} du_2 + \underbrace{\boxed{\frac{\partial \vec{r}}{\partial u_3}}}_{\vec{e}_{u_3}} du_3$$

base vectors

Definition: The BASE VECTORS $\vec{e}_{u_1}, \vec{e}_{u_2}, \vec{e}_{u_3}$ of the curvilinear coordinate system (u_1, u_2, u_3) are given through the derivatives of the RADIAL vector $\vec{r}$ as

$$\boxed{\vec{e}_{u_j} = \frac{\partial \vec{r}}{\partial u_j}},\ j = 1,2,3$$

From the above definition, we see that

$$d\vec{r} = \vec{e}_{u_1} du_1 + \vec{e}_{u_2} du_2 + \vec{e}_{u_3} du_3,$$

and

$$d^2s = d\vec{r} \cdot d\vec{r}$$

$$= (du_1, du_2, du_3)\underbrace{\begin{pmatrix} \vec{e}_{u_1}\cdot\vec{e}_{u_1} & \vec{e}_{u_1}\cdot\vec{e}_{u_2} & \vec{e}_{u_1}\cdot\vec{e}_{u_3} \\ \vec{e}_{u_2}\cdot\vec{e}_{u_1} & \vec{e}_{u_2}\cdot\vec{e}_{u_2} & \vec{e}_{u_2}\cdot\vec{e}_{u_3} \\ \vec{e}_{u_3}\cdot\vec{e}_{u_1} & \vec{e}_{u_3}\cdot\vec{e}_{u_2} & \vec{e}_{u_3}\cdot\vec{e}_{u_3} \end{pmatrix}}_{\text{METRIC MATRIX}}\begin{pmatrix} du_1 \\ du_2 \\ du_3 \end{pmatrix}$$

determines the curvature of the space.

* If a system is an orthogonal curvilinear system, the Metric matrix will be diagonal

Every metric system is orthogonal

$$= (du_1, du_2, du_3)\begin{pmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{pmatrix}\begin{pmatrix} du_1 \\ du_2 \\ du_3 \end{pmatrix}$$ (4)

$$g_{ij} = \vec{e}_{u_i} \cdot \vec{e}_{u_j}$$

Orthogonal Curvilinear Systems

If the basevectors $\vec{e}_{u_j}, j = 1,2,3$ are pairwise orthogonal, then the curvilinear coordinate system (u_1, u_2, u_3) is called ORTHOGONAL.

$$\vec{e}_{u_i} \cdot \vec{e}_{u_j} = \begin{cases} |\vec{e}_{u_i}|^2 & i = j \\ 0 & i \neq j \end{cases}$$

The metric matrix for these systems are DIAGONAL.

Scale Factors

The magnitudes h_{u_i} of the base vectors $\vec{e}_{u_i}$ of the curvilinear coordinate system (u_1, u_2, u_3) are called SCALE FACTORS

$$h_{u_1} = |\vec{e}_{u_1}|$$

$$h_{u_2} = |\vec{e}_{u_2}|$$

$$h_{u_3} = |\vec{e}_{u_3}|$$

Example 1

In this example we shall study the CYLINDRICAL POLAR SYSTEM OF COORDINATES (ρ, θ, z) defined through the coordinates (x, y, z) of the cartesian system as

$$\begin{cases} x = \rho\cos\theta, \\ y = \rho\sin\theta \\ z = z \end{cases} \quad \boxed{0 \le \theta < 2\pi;\ 0 < \rho < \infty; \\ -\infty < z < \infty}$$

$$\theta = \tan^{-1}\frac{y}{x}, \quad \rho = (x^2 + y^2)^{1/2}$$

Line Element

The line element for the cylindrical polar system is

$$d\vec{r} = \frac{\partial \vec{r}}{\partial \rho} d\rho + \frac{\partial \vec{r}}{\partial \theta} d\theta + \frac{\partial \vec{r}}{\partial z} dz$$

But

$$\vec{r} = x\underline{i} + y\underline{j} + z\underline{k}$$

$$= \rho(\cos\theta)\underline{i} + (\rho\sin\theta)\underline{j} + z\underline{k}$$

$$\frac{\partial \vec{r}}{\partial \rho} = \cos\theta\,\underline{i} + \sin\theta\,\underline{j}$$

$$\frac{\partial \vec{r}}{\partial \theta} = -\rho\sin\theta\,i + \rho\cos\theta\,j$$

$$\frac{\partial \vec{r}}{\partial z} = \underline{k}$$

~~Therefore~~

Volume Element

The volume element in the cartesian system is

(8a) $dx\,dy\,dz = dv$

(8b) $\left|\left(\frac{\partial x, \partial y, \partial z}{\partial u_1, \partial u_2, \partial u_3}\right)\right| du_1, du_2, du_3$

which the Jacobian

The expansion gives the volume element in the curvilinear system (u_1, u_2, u_3) through the Jacobian determinant

$$\left|\frac{\partial(x, y, z)}{\partial(u_1, u_2, u_3)}\right|$$

April 16, 2024

Cylindrical Polar System (ρ, θ, z)

$$\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \\ z = z \end{cases} \quad \begin{array}{l} 0 \le \rho < \infty \\ 0 \le \theta < 2\pi \\ -\infty < z < \infty \end{array} \quad \begin{array}{l} \theta = \tan^{-1} y/x \\ \rho = \sqrt{x^2+y^2} \end{array}$$

with reference to Cartesian (x, y, z)

$$d\vec{r} = \frac{\partial \vec{r}}{\partial \rho}d\rho + \frac{\partial \vec{r}}{\partial \theta}d\theta + \frac{\partial \vec{r}}{\partial z}dz$$

$$= \vec{e}_\rho d\rho + \vec{e}_\theta d\theta + \vec{e}_z dz$$

$$\vec{e}_\rho = \frac{\partial \vec{r}}{\partial \rho} = \cos\theta i + \sin\theta j$$

$$\vec{e}_\theta = \frac{\partial \vec{r}}{\partial \theta} = -\rho\sin\theta i + \rho\cos\theta j$$

$$\vec{e}_z = K$$

$$\vec{r} = xi + yj + zK$$

$$= \rho\cos\theta i + \rho\sin\theta j + zK$$

Elemental Arc Length

The elemental arc length ds is given through the relation

$$d^2s = d\vec{r}\cdot d\vec{r}$$

$$= (\vec{e}_\rho d\rho + \vec{e}_\theta d\theta + \vec{e}_z dz)\cdot(\vec{e}_\rho d\rho + \vec{e}_\theta d\theta + \vec{e}_z dz)$$

$$= (\vec{e}_\rho\cdot\vec{e}_\rho)d^2\rho + (\vec{e}_\theta\cdot\vec{e}_\theta)d^2\theta + (\vec{e}_z\cdot\vec{e}_z)d^2z + 2(\vec{e}_\rho\cdot\vec{e}_\theta)d\rho d\theta + 2(\vec{e}_\rho\cdot\vec{e}_z)d\rho dz + 2(\vec{e}_\theta\cdot\vec{e}_z)d\theta dz$$

Since

$$(3)\begin{cases} \vec{e}_\rho\cdot\vec{e}_\theta = -\rho\cos\theta\sin\theta + \rho\sin\theta\cos\theta = 0 \\ \vec{e}_\rho\cdot\vec{e}_z = 0 \\ \vec{e}_\theta\cdot\vec{e}_z = 0 \end{cases}$$

The system is ORTHOGONAL

Therefore,

$$d^2s = |\vec{e}_\rho|^2 d^2\rho + |\vec{e}_\theta|^2 d\theta^2 + |\vec{e}_z|^2 dz$$

$$= d\rho^2 + \rho^2 d\theta^2 + dz^2$$

$$ds = \sqrt{d\rho^2 + \rho^2 d\theta^2 + dz^2}$$

Elemental Volume

The elemental volume in the Cartesian system (x, y, z) is given by

$$dV = dx\,dy\,dz \quad \text{---} (5)$$

In the 3-D cylindrical polar system is

$$dV = \left|\frac{\partial(x,y,z)}{\partial(\rho,\theta,z)}\right| d\rho\, d\theta\, dz$$

$$= \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{vmatrix} d\rho\, d\theta\, dz \qquad \begin{array}{l} x = \rho\cos\theta \\ y = \rho\sin\theta \\ z = z \end{array}$$

$$= \begin{vmatrix} \cos\theta & -\rho\sin\theta & 0 \\ \sin\theta & \rho\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} d\rho\, d\theta\, dz$$

$$= \begin{vmatrix} \cos\theta & -\rho\sin\theta \\ \sin\theta & \rho\cos\theta \end{vmatrix} d\rho\, d\theta\, dz$$

$$dV = \rho\, d\rho\, d\theta\, dz \quad \text{----} (6)$$

Eqn (6) is the elemental volume of a polar cylindrical system.

Coordinate Surface

The coordinat surfaces of the CYLINDRICAL POLAR system (ρ, θ, z) are

$$(7)\ \begin{cases} \rho = \text{const} \\ \theta = \text{const} \\ z = \text{const} \end{cases}$$

(7a), That is, ρ = const, corresponds to the curved surface of the cylinder;

7b), i.e, θ = const, corresponds to a half plane; an

7c, i.e, z = const, corresponds to a plane.

CALCULUS IN OTHOGONAL COORDINATE SYSTEM

We shall define the HAMILTONIAN ∇ in an ORTHOGONAL curvilinear system (u_1, u_2, u_3) as

$$\nabla = \frac{\hat{e}_{u_1}}{h_{u_1}}\frac{\partial}{\partial u_1} + \frac{\hat{e}_{u_2}}{h_{u_2}}\frac{\partial}{\partial u_2} + \frac{\hat{e}_{u_3}}{h_{u_3}}\frac{\partial}{\partial u_3}$$

Example: In the cylindrical polar system (ρ, θ, z);

$$h_\rho = |\vec{e}_\rho| = |\cos\theta i + \sin\theta j| = 1$$

$$h_\theta = |\vec{e}_\theta| = |-\rho\sin\theta i + \rho\cos\theta j| = \rho$$

$$h_z = |\vec{e}_z| = |k| = 1$$

Example 2 $(u_1 = \rho,\ u_2 = \theta,\ u_3 = z)$

Now in the cylindrical polar system

$$\nabla = \hat{e}_\rho\frac{\partial}{\partial \rho} + \frac{\hat{e}_\theta}{\rho}\frac{\partial}{\partial \theta} + \hat{e}_z\frac{\partial}{\partial z}$$

Example 5

Calculate

(i) ∇u_1 (ii) ∇u_2 (iii) ∇u_3 (iv) $\nabla\cdot\left(\frac{\hat{e}_{u_3}}{h_{u_1}h_{u_2}}\right)$

Solution

(i) $$\nabla u_1 = \frac{\hat{e}_{u_1}}{h_{u_1}}\frac{\partial u_1}{\partial u_1} + \frac{\hat{e}_{u_2}}{h_{u_2}}\frac{\partial u_1}{\partial u_2} + \frac{\hat{e}_{u_3}}{h_{u_3}}\frac{\partial u_1}{\partial u_3}$$

$$= \frac{\hat{e}_{u_1}}{h_{u_1}}\times 1 + \frac{\hat{e}_{u_2}}{h_{u_2}}\times 0 + \frac{\hat{e}_{u_3}}{h_{u_3}}\times 0$$

$$= \frac{\hat{e}_{u_1}}{h_{u_1}}$$ //

(ii) In the same fashion $\nabla u_2 = \frac{\hat{e}_{u_2}}{h_{u_2}}$

(iii) $\nabla u_3 = \frac{\hat{e}_{u_3}}{h_{u_3}}$

(iv) $$\nabla u_1 \times \nabla u_2 = \frac{\hat{e}_{u_1}}{h_{u_1}} \times \frac{\hat{e}_{u_2}}{h_{u_2}}$$

$$= \frac{1}{h_{u_1}h_{u_2}}\hat{e}_{u_1}\times\hat{e}_{u_2} = \frac{1}{h_{u_1}h_{u_2}}\hat{e}_{u_3}$$

$$\therefore \nabla\cdot\left(\frac{1}{h_{u_1}h_{u_2}}\hat{e}_{u_3}\right) = \nabla\cdot(\nabla u_1 \times \nabla u_2)$$

$$\boxed{\nabla\cdot(\vec{A}\times\vec{B}) = \vec{B}\cdot(\nabla\times\vec{A}) - \vec{A}\cdot(\nabla\times\vec{B})}$$

$$\to \nabla u_2\cdot(\nabla\times\nabla u_1) - \nabla u_1\cdot(\nabla\times\nabla u_2)$$

$$= \nabla u_2\cdot\vec{0} - \nabla u_1\cdot\vec{0}$$

$$= 0.$$

$\hat{e}_{u_2} \times \hat{e}_{u_1} = e$

17th April, 2024

Example 1

Calculate (i) $\nabla \rho$ (ii) $\nabla(1/\rho)$, $\rho > 0$

(iii) $\nabla(\log_e \rho)$. $\rho > 0$

solution

(i) $\nabla \rho = \hat{e}_\rho \frac{\partial \rho}{\partial \rho} + \frac{\hat{e}_\theta}{\rho} \frac{\partial \rho}{\partial \theta} + \hat{e}_z \frac{\partial \rho}{\partial z}$

$= \hat{e}_\rho$

(ii) $\nabla\left(\frac{1}{\rho}\right) = \hat{e}_\rho \frac{\partial}{\partial \rho}\left(\frac{1}{\rho}\right) = -\rho^{-2}\hat{e}_\rho$

(iii) $\nabla(\log_e \rho) = \frac{\hat{e}_\rho}{\rho}$

From the last class, we observed that for an orthogonal curvilinear system (u_1, u_2, u_3), and unit base vectors $\hat{e}_{u_1}, \hat{e}_{u_2}, \hat{e}_{u_3}$,

$$\nabla \cdot \left(\frac{\hat{e}_{u_3}}{h_{u_1} h_{u_2}}\right) = 0 = \nabla \cdot \left(\frac{\hat{e}_{u_1}}{h_{u_2} h_{u_3}}\right) = 0 = \nabla\left(\frac{\hat{e}_{u_2}}{h_{u_1} h_{u_3}}\right)$$

Divergence

The divergence $\nabla \cdot \vec{A}$ of a vector field $\vec{A}(u_1, u_2, u_3)$ in an orthogonal curvilinear system is constructed below

Let

$\vec{A} = A_1 \hat{e}_{u_1} + A_2 \hat{e}_{u_2} + A_3 \hat{e}_{u_3}$,

$A_1 = A_1(u_1, u_2, u_3)$; $A_2 = A_2(u_1, u_2, u)$ &

$A_3 = A_3(u_1, u_2, u_3)$

Then $\nabla \cdot \vec{A} = \nabla \cdot (A_1 \hat{e}_{u_1}) + \nabla \cdot (A_2 \hat{e}_{u_2})$

$+ \nabla \cdot (A_3 \hat{e}_{u_3})$ ------ (1)

Consider

$\nabla \cdot (A_1 \hat{e}_{u_1})$

$$\nabla \cdot (A_1 \hat{e}_{u_1}) = \nabla \cdot \left[\underbrace{(A_1 h_{u_2} h_{u_3})}_{\phi} \underbrace{\left(\frac{\hat{e}_{u_1}}{h_{u_2} h_{u_3}}\right)}_{\vec{V}}\right]$$

$= \nabla \cdot (\phi \vec{V}) = \left(\vec{V} (\nabla \phi) + \phi (\nabla \cdot \vec{V})\right.$

$= \frac{\hat{e}_{u_1}}{h_{u_2} h_{u_3}} \cdot \nabla(A_1 h_{u_2} h_{u_3}) + A_1 h_{u_2} h_{u_3} \nabla \cdot \left(\frac{\hat{e}_{u_1}}{h_{u_2} h_{u_3}}\right)$

$= \frac{\hat{e}_{u_1}}{h_{u_2} h_{u_3}} \cdot \nabla(A_1 h_{u_2} h_{u_3}) = \frac{\hat{e}_{u_1}}{h_{u_1} h_{u_3}} \cdot \left\{\frac{\hat{e}_{u_1}}{h_{u_1}} \frac{\partial}{\partial u_1}(A_1 h_{u_2} h_{u_3}) \right.$

$\left. + \frac{\hat{e}_{u_2}}{h_{u_2}} \frac{\partial}{\partial u_2}(A_1 h_{u_2} h_{u_3}) + \frac{\hat{e}_{u_3}}{h_{u_3}} \frac{\partial}{\partial u_3}(A_1 h_{u_2} h_{u_3})\right)$

$= \frac{1}{h_{u_1} h_{u_2} h_{u_3}} \frac{\partial}{\partial u_1}(A_1 h_{u_2} h_{u_3}) = \nabla \cdot (A_1 \hat{e}_{u_1})$

$\frac{1}{h_{u_1} h_{u_2} h_{u_3}} \frac{\partial}{\partial u_2}(A_2 h_{u_1} h_{u_3}) = \nabla \cdot (A_2 \hat{e}_{u_2})$

$\frac{1}{h_{u_1} h_{u_2} h_{u_3}} \frac{\partial}{\partial u_3}(A_3 h_{u_1} h_{u_2}) = \nabla \cdot (A_3 \hat{e}_{u_3})$

$$\therefore \nabla \cdot \vec{A} = \frac{1}{h_{u_1} h_{u_2} h_{u_3}} \left\{\frac{\partial}{\partial u_1}(A_1 h_{u_2} h_{u_3}) + \frac{\partial}{\partial u_2}(A_2 h_{u_1} h_{u_3}) + \frac{\partial}{\partial u_3}(A_3 h_{u_1} h_{u_2})\right\} \quad \text{---(2)}$$

Example 2: Write out the divergence $\nabla \cdot \vec{A}$ of a vector field $\vec{A}$ in the cylindrical polar system (ρ, θ, z)

solution

$h_\rho = 1$, $h_\theta = \rho$, $h_z = 1$; $u_1 = \rho$, $u_2 = \theta$, $u_3 = z$

Therefore, by eq. 2 Let $\vec{A} = A_1 \hat{e}_\rho + A_2 \hat{e}_\theta + A_3 \hat{e}_z$

$$\nabla \cdot \vec{A} = \frac{1}{h_\rho h_\theta h_z} \left\{\frac{\partial}{\partial \rho}(A_1 h_\theta h_z) + \frac{\partial}{\partial \theta}(A_2 h_\rho h_z) + \frac{\partial}{\partial z}(A_3 h_\rho h_\theta)\right.$$

$$= \frac{1}{\rho} \left\{\frac{\partial}{\partial \rho}(\rho A_1) + \frac{\partial A_2}{\partial \theta} + \frac{\partial}{\partial z}(\rho A_3)\right\}$$

$$= \frac{1}{\rho}\frac{\partial}{\partial \rho}\rho A_1 + \frac{1}{\rho}\frac{\partial A_2}{\partial \theta} + \frac{\partial A_3}{\partial z}$$

THE LAPLACIAN

From the foregoing, we know that the Laplacian of a scalar field $\Phi(u_1, u_2, u_3)$ is constructed as

$$\nabla^2 \Phi = \nabla \cdot (\nabla \Phi)$$

$$\nabla \cdot \left(i\frac{\partial \Phi}{\partial x} + j\frac{\partial \Phi}{\partial x} + k\frac{\partial \Phi}{\partial z} \right)$$

$$= \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2}$$

$$= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \Phi$$

$$\nabla \Phi = \frac{\hat{e}_{u_1}}{h_{u_1}}\frac{\partial \Phi}{\partial u_1} + \frac{\hat{e}_{u_2}}{h_{u_2}}\frac{\partial \Phi}{\partial u_2} + \frac{\hat{e}_{u_3}}{h_{u_3}}\frac{\partial \Phi}{\partial u_3}$$

$$= \hat{e}_{u_1}\left(\frac{1}{h_{u_1}}\frac{\partial \Phi}{\partial u_1}\right) + \hat{e}_{u_2}\left(\frac{1}{h_{u_2}}\frac{\partial \Phi}{\partial u_2}\right) + \hat{e}_{u_3}\left(\frac{1}{h_{u_3}}\frac{\partial \Phi}{\partial u_3}\right) \quad \text{...... (3)}$$

$$\nabla \cdot \vec{A} = \frac{1}{h_{u_1}h_{u_2}h_{u_3}} \left\{ \frac{\partial}{\partial u_1}(A_1 h_{u_2} h_{u_3}) + \frac{\partial}{\partial u_2}(A_2 h_{u_1} h_{u_3}) + \frac{\partial}{\partial u_3}(A_3 h_{u_1} h_{u_2}) \right\}$$

$$\nabla^2 \Phi = \nabla \cdot (\nabla \Phi)$$

$$= \frac{1}{h_{u_1}h_{u_2}h_{u_3}} \left\{ \frac{\partial}{\partial u_1}\left(\frac{1}{h_{u_1}}\frac{\partial \Phi}{\partial u_1} h_{u_2}h_{u_3}\right) + \frac{\partial}{\partial u_2}\left(\frac{1}{h_{u_2}}\frac{\partial \Phi}{\partial u_2} h_{u_1}h_{u_3}\right) + \frac{\partial}{\partial u_3}\left(\frac{1}{h_{u_3}}\frac{\partial \Phi}{\partial u_3} h_{u_1}h_{u_2}\right) \right\}$$

$$= \frac{1}{h_{u_1}h_{u_2}h_{u_3}} \left[\frac{\partial}{\partial u_1}\left(\frac{h_{u_2}h_{u_3}}{h_{u_1}}\frac{\partial \Phi}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_{u_1}h_{u_3}}{h_{u_2}}\frac{\partial \Phi}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{h_{u_1}h_{u_2}}{h_{u_3}}\frac{\partial \Phi}{\partial u_3}\right) \right]$$

Example 3

Write out the Laplacian $\nabla^2 \Phi$ of a scalar field $\Phi(\rho, \theta, z)$ in the cylindrical polar system (ρ, θ, z)

Solution

$h_\rho = 1$, $h_\theta = \rho$, $h_z = 1$; $u_1 = \rho$, $u_2 = \theta$, $u_3 = z$

Therefore by equation 4

$$\nabla^2 \Phi = \frac{1}{\rho}\left[\frac{\partial}{\partial \rho}\left(\rho\frac{\partial \Phi}{\partial \rho}\right) + \frac{\partial}{\partial \theta}\left(\frac{1}{\rho}\frac{\partial \Phi}{\partial \theta}\right) + \frac{\partial}{\partial z}\left(\rho\frac{\partial \Phi}{\partial z}\right)\right]$$

$$= \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\frac{1}{\rho}\frac{\partial \Phi}{\partial \rho}\right) + \frac{1}{\rho^2}\left(\frac{\partial \Phi}{\partial \theta}\right) + \frac{\partial^2 \Phi}{\partial z^2} \quad \text{...... (*)}$$

16th April, 2024

Example 4: calculate

(i) $\nabla^2 \log \rho$ (ii) $\nabla^2(\rho^{-k})$, $\rho > 0$ in the general polar system (ρ, θ, z)

Solution

(i) Using eqn (*)

$$\nabla^2 \log_e \rho = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial \log_e \rho}{\partial \rho}\right) + \frac{1}{\rho^2}\underbrace{\frac{\partial^2 \log_e \rho}{\partial \theta^2}}_{0} + \underbrace{\frac{\partial^2 \log_e \rho}{\partial z^2}}_{0}$$

$\equiv \ln \rho$

$$= \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \times \frac{1}{\rho}\right) = 0$$

k is a real number not equal to 0.

(ii) $\nabla^2(\rho^{-k})$, $k \in \mathbb{R}\setminus\{0\}$

$$= \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\rho^{-k}}{\partial\rho}\right) = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(-k\rho^{-k}\right)$$

$$= k^2\rho^{-k-2}$$

Rotor/Curl of a Vector

Let $\vec{A}$ be a vector field in an orthogonal curvilinear system (u_1, u_2, u_3). Then the curl $(\nabla\times\vec{A})$ of $\vec{A}$ is given as

$$\nabla\times\vec{A}(u_1,u_2,u_3) = \nabla\times(A_1\vec{e}_{u_1} + A_2\vec{e}_{u_2} + A_3\vec{e}_{u_3})$$

$$= \nabla\times(A_1\vec{e}_{u_1}) + \nabla\times(A_2\vec{e}_{u_2}) + \nabla\times(A_3\vec{e}_{u_3}) \quad \ldots (2)$$

Exercise: Show that $\nabla\times\left(\frac{\hat{e}_{u_1}}{h_{u_1}}\right) = \vec{0}$ ...(3)

Solution to Exercise)

$\nabla u_1 = ?$

$$\nabla u_1 = \frac{\hat{e}_1}{h_{u_1}}\frac{\partial u_1}{\partial u_1} + \cancel{\frac{\hat{e}_2}{h_{u_2}}\frac{\partial u_1}{\partial u_2}} + \cancel{\frac{\hat{e}_3}{h_{u_3}}\frac{\partial u_1}{\partial u_3}}$$

$$\nabla u_1 = \frac{\hat{e}_1}{h_{u_1}}$$

$$\nabla\times\left(\frac{\hat{e}_1}{h_{u_1}}\right) = \nabla\times(\nabla u_1) = \vec{0}$$

Recall that $\nabla\cdot(\Phi\vec{V}) = \vec{V}\cdot\nabla\Phi + \Phi\nabla\cdot\vec{V}$

Now, one can write $\nabla\times(A_1\hat{e}_{u_1})$ as

$$\nabla\times\left(A_1 h_{u_1}\left(\frac{\hat{e}_{u_1}}{h_{u_1}}\right)\right) = \nabla(A_1 h_{u_1})\times\frac{\hat{e}_{u_1}}{h_{u_1}} + A_1 h_{u_1}\underbrace{\nabla\times\left(\frac{\hat{e}_{u_1}}{h_{u_1}}\right)}_{\vec{0}}$$

$$= \nabla(A_1 h_{u_1})\times\frac{\hat{e}_{u_1}}{h_{u_1}}$$

$$= \left(\frac{\hat{e}_{u_1}}{h_{u_1}}\frac{\partial}{\partial u_1}(A_1h_{u_1}) + \frac{\hat{e}_{u_2}}{h_{u_2}}\frac{\partial}{\partial u_2}(A_1h_{u_1}) + \frac{\hat{e}_{u_3}}{h_{u_3}}\frac{\partial}{\partial u_3}(A_1h_{u_1})\right)\times\frac{\hat{e}_{u_1}}{\boxed{h_{u_1}}}$$

$$= -\frac{\hat{e}_{u_3}}{h_{u_1}h_{u_2}}\frac{\partial}{\partial u_3}(A_1h_{u_1}) + \frac{\hat{e}_{u_2}}{h_{u_3}h_{u_1}}\frac{\partial}{\partial u_3}(A_1h_{u_1}) \quad \ldots (I)$$

$$\nabla\times(A_2\hat{e}_{u_2}) = \nabla(A_2h_{u_2})\times\hat{e}_{u_2}/h_{u_2}$$

$$= \left(\frac{\hat{e}_{u_1}}{h_{u_1}}\frac{\partial}{\partial u_1}(A_2h_{u_2}) + \frac{\hat{e}_{u_3}}{h_{u_3}}\frac{\partial}{\partial u_3}(A_2h_{u_2})\right)\times\frac{\hat{e}_{u_2}}{h_{u_2}}$$

$$= \frac{\hat{e}_{u_3}}{h_{u_1}h_{u_2}}\frac{\partial}{\partial u_1}(A_2h_{u_2}) - \frac{\hat{e}_{u_1}}{h_{u_3}h_{u_2}}\frac{\partial}{\partial u_3}(A_2h_{u_2}) \quad \ldots (II)$$

$$\nabla\times(A_3\hat{e}_{u_3}) = \nabla(A_3h_{u_3})\times\frac{\hat{e}_{u_3}}{h_{u_3}}$$

$$= \left\{\frac{\hat{e}_{u_1}}{h_{u_1}}\frac{\partial}{\partial u_1}(A_3h_{u_3}) + \frac{\hat{e}_{u_2}}{h_{u_2}}\frac{\partial}{\partial u_2}(A_3h_{u_3})\right\}\times\frac{\hat{e}_{u_3}}{h_{u_3}}$$

$$= \frac{\hat{e}_{u_2}}{h_{u_1}h_{u_3}}\frac{\partial}{\partial u_1}(A_3h_{u_3}) + \frac{\hat{e}_{u_1}}{h_{u_2}h_{u_3}}\frac{\partial}{\partial u_2}(A_3h_{u_3}) \quad \ldots (III)$$

$$\nabla\times\vec{A} = \text{eqn}(I + II + III)$$

$$= \frac{\hat{e}_{u_1}}{h_{u_2}h_{u_3}}\left(\frac{\partial}{\partial u_2}(A_3h_{u_3}) - \frac{\partial}{\partial u_3}(A_2h_{u_2})\right)$$

$$+ \frac{\hat{e}_{u_2}}{h_{u_1}h_{u_3}}\left(\frac{\partial}{\partial u_3}(A_1h_{u_1}) - \frac{\partial}{\partial u_1}(A_3h_{u_3})\right)$$

$$+ \frac{\hat{e}_{u_3}}{h_{u_1}h_{u_2}}\left(\frac{\partial(A_2h_{u_2})}{\partial u_1} - \frac{\partial}{\partial u_2}(A_1h_{u_1})\right) \quad \ldots (4)$$

$$\nabla\times\vec{A} = \begin{vmatrix} \frac{\hat{e}_{u_1}}{h_{u_2}h_{u_3}} & \frac{\hat{e}_{u_2}}{h_{u_1}h_{u_3}} & \frac{\hat{e}_{u_3}}{h_{u_1}h_{u_2}} \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ A_1h_{u_1} & A_2h_{u_2} & A_3h_{u_3} \end{vmatrix}$$

$$\begin{vmatrix} \frac{\hat{e}_\rho}{\rho} & \hat{e}_\theta & \frac{\hat{e}_z}{\rho} \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial z} \\ A_1 & A_2\rho & A_3 \end{vmatrix}$$

19th April, 2024

Example 1

Calculate

(i) $\nabla\times\vec{r}$ (ii) $\nabla\times\hat{e}_\theta$ (iii) $\nabla\times\hat{e}_x$ in the cylindrical polar system (ρ, θ, z)

Solution

In the cylindrical polar coordinate system (ρ,θ,z)

If $\vec{A} = A_1\hat{e}_\rho + A_2\hat{e}_\theta + A_3\hat{e}_z$

then

$$\nabla\times\vec{A} = \begin{vmatrix} \frac{\hat{e}_\rho}{h_\theta h_z} & \frac{\hat{e}_\theta}{h_\rho h_z} & \frac{\hat{e}_z}{h_\rho h_\theta} \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial z} \\ h_\rho A_1 & h_\theta A_2 & h_z A_3 \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\hat{e}_\rho}{\rho} & \hat{e}_\theta & \hat{e}_z \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial z} \\ A_1 & \rho A_2 & A_3 \end{vmatrix}$$

$\vec{r} = \rho\hat{e}_\rho = \rho\hat{e}_\rho + 0\hat{e}_\theta + 0\hat{e}_z$

$$\nabla\times\vec{r} = \begin{vmatrix} \hat{e}_\rho & \hat{e}_\theta & \frac{\hat{e}_z}{\rho} \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial z} \\ \rho & 0 & 0 \end{vmatrix}$$

$$= \frac{\hat{e}_\rho}{\rho}\left[\frac{\partial 0}{\partial\theta} - \frac{\partial 0}{\partial z}\right] - \hat{e}_\theta\left[\frac{\partial 0}{\partial\rho} - \frac{\partial\rho}{\partial z}\right] + \frac{\hat{e}_z}{\rho}\left[\frac{\partial 0}{\partial\rho} - \frac{\partial\rho}{\partial\theta}\right] = \vec{0}$$

(ii) $$\nabla\times\hat{e}_\theta = \begin{vmatrix} \frac{\hat{e}_\rho}{\rho} & \hat{e}_\theta & \frac{\hat{e}_z}{\rho} \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial z} \\ 0 & \rho\cdot 1 & 0 \end{vmatrix}$$

$\hat{e}_\theta = 0\cdot\hat{e}_\rho + 1\cdot\hat{e}_\theta + 0\cdot\hat{e}_z$

$$= \frac{\hat{e}_\rho}{\rho}\left[\frac{\partial 0}{\partial\theta} - \frac{\partial\rho}{\partial z}\right] + \hat{e}_\theta\left[\frac{\partial 0}{\partial z} - \frac{\partial 0}{\partial\rho}\right]$$

$$+ \frac{\hat{e}_z}{\rho}\left[\frac{\partial\rho}{\partial\rho} - \frac{\partial 0}{\partial\theta}\right]$$

$$= \frac{\hat{e}_z}{\rho}$$

Spherical polar System (ρ, θ, ψ)

The spherical polar system (ρ, θ, ψ) is defined by the transformations

$$\begin{cases} x = \rho\sin\theta\cos\psi \\ y = \rho\sin\theta\sin\psi \\ z = \rho\cos\theta \end{cases}, \quad \begin{array}{l} 0 \le \rho < \infty \\ 0 \le \theta \le \pi \\ 0 \le \psi < 2\pi \end{array}$$

$\boxed{\text{Comment}}$: The scale factors h_ρ, h_θ, h_ψ of the Spherical polar system (ρ, θ, ψ) are

$$\begin{cases} h_\rho = 1 \\ h_\theta = \rho \\ h_\psi = \rho\sin\theta \end{cases}$$

$$\nabla f = \frac{\hat{e}_\rho}{h_\rho}\frac{\partial f}{\partial\rho} + \frac{\hat{e}_\theta}{h_\theta}\frac{\partial f}{\partial\theta} + \frac{\hat{e}_\psi}{h_\psi}\frac{\partial f}{\partial\psi}, \quad f = C(\mathbb{R}^3)$$

$$= \hat{e}_\rho\frac{\partial f}{\partial\rho} + \frac{\hat{e}_\theta}{\rho}\frac{\partial f}{\partial\theta} + \frac{e_\psi}{\rho\sin\theta}\frac{\partial f}{\partial\psi}$$

Let $A = A_1\hat{e}_\rho + A_2\hat{e}_\theta + A_3\hat{e}_\psi$

$$\nabla\cdot\vec{A} = \frac{1}{\rho\sin\theta}\left\{\hat{e}_\rho\frac{\partial}{\partial\rho}(\rho\sin\theta A_1) + \hat{e}_\theta\frac{\partial}{\partial\theta}(\rho\sin\theta A_2) + \hat{e}_\psi\frac{\partial}{\partial\psi}(\rho\sin\theta A_3)\right)$$

$$\nabla^2 f = \frac{1}{h_{u_1}h_{u_2}h_{u_3}}\left[\frac{\partial}{\partial u_1}\left(\frac{h_{u_2}h_{u_3}}{h_{u_1}}\frac{\partial f}{\partial u_1}\right) + \dots\right.$$

$$= \frac{1}{\rho\sin\theta}\left[\frac{\partial}{\partial\rho}\left(\rho^2\sin\theta\frac{\partial f}{\partial\rho}\right) + \frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{\partial}{\partial\psi}\left(\frac{1}{\sin\theta}\frac{\partial f}{\partial\psi}\right)\right]$$

$(1,0,0) = i \neq \underline{i}(x,y,z)$
$(0,1,0) = j \neq \underline{j}(x,y,z)$
$(0,0,1) = k \neq \underline{k}(x,y,z)$

$$\nabla \times \vec{A} = \begin{vmatrix} \frac{\hat{e}_\rho}{h_\theta h_\psi} & \frac{\hat{e}_\theta}{h_\rho h_\psi} & \frac{\hat{e}_\psi}{h_\rho h_\theta} \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \psi} \\ h_\rho A_1 & h_\theta A_2 & h_\psi A_3 \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\hat{e}_\rho}{\rho \sin\theta} & \frac{\hat{e}_\theta}{\rho \sin\theta} & \frac{\hat{e}_\psi}{\rho} \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \psi} \\ A_1 & \rho A_2 & \rho \sin\theta A_3 \end{vmatrix}$$

Example 2: ~~Calculate~~

Let $\vec{v} = f(\rho)\hat{e}_z$. Calculate

(i) $\nabla \times \vec{v}$, (ii) $\vec{v} \times (\nabla \times \vec{v})$

(iii) $\nabla \times (\vec{v} \times (\nabla \times \vec{v}))$

in

(a) The cylindrical polar system (ρ, θ, z)

(b) The spherical polar system (ρ, θ, ψ)

Solution

$\vec{v} = 0\,\hat{e}_\rho + 0\,\hat{e}_\theta + f(\rho)\hat{e}_z$

$$\nabla \times \vec{v} = \begin{vmatrix} \frac{\hat{e}_\rho}{\rho} & \hat{e}_\theta & \frac{\hat{e}_z}{\rho} \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ f(\rho) & 0 & 0 \end{vmatrix}$$

$$= \frac{\hat{e}_\rho}{\rho}\begin{vmatrix} \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ 0 & f(\rho) \end{vmatrix} - \hat{e}_\theta \begin{vmatrix} \frac{\partial}{\partial \rho} & \frac{\partial}{\partial z} \\ 0 & f(\rho) \end{vmatrix} + \frac{\hat{e}_z}{\rho}\begin{vmatrix} \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \theta} \\ 0 & 0 \end{vmatrix}$$

$$= -e_\theta \frac{\partial f}{\partial \rho}$$

(ii) $\vec{v} \times (\nabla \times \vec{v}) = (f(\rho)\,\hat{e}_z) \times \left(-\hat{e}_\theta \frac{\partial f}{\partial \rho}\right)$

$$= \begin{vmatrix} \hat{e}_\rho & \hat{e}_\theta & \hat{e}_z \\ 0 & 0 & f(\rho) \\ 0 & -\frac{\partial f}{\partial \rho} & 0 \end{vmatrix}$$

$$= \hat{e}_\rho \left(f \frac{\partial f}{\partial \rho}\right)$$

(iii) $\nabla \times (v \times (\nabla \times \vec{v})) = \nabla \times \left(f \frac{\partial f}{\partial \rho}\,\hat{e}_\rho\right)$

$$= \begin{vmatrix} \frac{\hat{e}_\rho}{\rho} & \hat{e}_\theta & \frac{\hat{e}_z}{\rho} \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ f\frac{\partial f}{\partial \rho} & 0 & 0 \end{vmatrix} = \vec{0}$$

2nd April, 2024

Vector Laplacian

In the 3D Cartesian system (x, y, z), the vector Laplacian $\nabla^2 \vec{A}$ of a vector field $\vec{A}$ is given by

$$\nabla^2 \vec{A} = \nabla^2 (A_1(x,y,z)\,i + A_2(x,y,z)\,j + A_3(x,y,z)\,k)$$

$$= \nabla^2 (A_1 i) + \nabla^2 (A_2 j) + \nabla^2 (A_3 k)$$

$$= (\nabla^2 A_1)\underline{i} + (\nabla^2 A_2)\underline{j} + (\nabla^2 A_3)\underline{k}$$

Recall the vector identity

$$\nabla \times (\nabla \times \vec{A}) = \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A} \quad \ldots\ldots (1)$$

$$\therefore \nabla^2 \vec{A} = \nabla(\nabla \cdot \vec{A}) - \nabla \times (\nabla \times \vec{A}) \quad \ldots (2)$$

We shall employ Eqn 1 and Eqn 2 in the calculation of vector Laplacian of vector fields in orthogonal curvilinear systems since their base vectors are vector fields.

Example: Calculate the vector Laplacian $\nabla^2 e_\theta$ in the cylindrical polar system (ρ, θ, z)

Solution

We shall employ identity (2)

$$\nabla^2 \hat e_\theta = \nabla(\underbrace{\nabla\cdot \hat e_\theta}_{(a)}) - \nabla\times(\underbrace{\nabla\times \hat e_\theta}_{(b)})$$

(a) $\nabla\cdot\hat e_\theta = \nabla\cdot(0.\hat e_\rho + 1.\hat e_\theta + 0.e_z)$

$$= \frac{1}{\rho}\left[\frac{\partial}{\partial\rho}(0.h_\theta h_z) + \frac{\partial}{\partial\theta}(1\cdot h_\rho h_z) + \frac{\partial}{\partial z}(0.h_\rho h_\theta)\right]$$

$$= 0 \ldots\ldots (*)$$

(b) $\nabla\times\hat e_\theta = \nabla\times(0.\hat e_\rho + 1\cdot\hat e_\theta + 0\cdot\hat e_z)$

$$= \begin{vmatrix} \frac{\hat e_\rho}{\rho} & \hat e_\theta & \frac{\hat e_z}{\rho} \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial z} \\ 0\cdot 1 & 1\cdot\rho & 0\cdot 1 \end{vmatrix}$$

$$= \frac{\hat e_\rho}{\rho}(0) + \hat e_\theta(0) + \frac{\hat e_z}{\rho}\left(\frac{\partial\rho}{\partial\rho} - 0\right)$$

$$= \frac{\hat e_z}{\rho}$$

Now

$$\nabla\times(\nabla\times\hat e_\theta) = \nabla\times\left(\frac{\hat e_z}{\rho}\right) \pm \nabla\times\left(0.e_\rho + 0.\hat e_\theta + \frac{1}{\rho}\hat e_z\right)$$

$$= \begin{vmatrix} \frac{\hat e_\rho}{\rho} & \hat e_\theta & \frac{\hat e_z}{\rho} \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial z} \\ 0\cdot 1 & 0\cdot\rho & \frac{1}{\rho}\cdot 1 \end{vmatrix}$$

$$= \frac{\hat e_\rho}{\rho}\left(\frac{\partial}{\partial\theta}\left(\frac{1}{\rho}\right) - 0\right) + \hat e_\theta\left(\frac{\partial 0}{\partial z} - \frac{\partial}{\partial\rho}\left(\frac{1}{\rho}\right)\right)$$

$$+ \frac{\hat e_z}{\rho}\left(\frac{\partial 0}{\partial\rho} - \frac{\partial 0}{\partial\theta}\right) = \frac{1}{\rho^2}\hat e_\theta$$

$$\therefore \nabla^2\hat e_\theta = \nabla(0) - \underbrace{\frac{1}{\rho^2}\hat e_\theta}_{\nabla\times(\nabla\times\hat e_\theta)} = -\frac{1}{\rho^2}\hat e_\theta$$

Pure radial

Example 2: Calculate $\nabla^2(f(\rho)\hat e_\rho)$ in the (i) cylindrical polar system (ρ, θ, z) (ii) spherical polar system (ρ, θ, ψ)

We know that

$$\nabla^2(f(\rho)\hat e_\rho) = \underbrace{\nabla(\nabla\cdot f(\rho)\hat e_\rho)}_{(a)} - \underbrace{\nabla\times(\nabla\times f(\rho)\hat e_\rho)}_{(b)}$$

(i) $\nabla\cdot(f(\rho)\hat e_\rho) = \nabla\cdot(f(\rho)e_\rho + 0\hat e_\theta + 0\hat e_z)$

$$= \frac{1}{h_\rho h_\theta h_z}\left[\frac{\partial}{\partial\rho}(h_\theta h_z f(\rho))\right] = \frac{1}{\rho}\left[\frac{\partial}{\partial\rho}(\rho f(\rho))\right]$$

$$= \frac{1}{\rho}\left[f(\rho) + \rho f'(\rho)\right] = \frac{f(\rho)}{\rho} + f'(\rho)$$

$$\nabla(\nabla\cdot f(\rho)\hat e_\rho) = \frac{\hat e_\rho}{1}\frac{\partial}{\partial\rho}\left(\frac{f(\rho)}{\rho} + f'(\rho)\right) +$$

$$\frac{\hat e_\theta}{h_\theta}\frac{\partial}{\partial\theta}\left(\frac{f(\rho)}{\rho} + f'(\rho)\right) + \frac{\hat e_z}{h_z}\frac{\partial}{\partial z}\left(\frac{f(\rho)}{\rho} + f'(\rho)\right)$$

$$= \hat e_\rho\left[f''(\rho) + \frac{f'(\rho)}{\rho} - \frac{f}{\rho^2}\right] \ldots\ldots \#$$

(b) $\nabla\times f(\rho)\hat e_\rho$

$$= \begin{vmatrix} \frac{\hat e_\rho}{\rho} & \hat e_\theta & \hat e_z \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial z} \\ f(\rho) & 0 & 0 \end{vmatrix}$$

$$= \vec 0 \ldots\ldots \#$$

From eq (+) & (++), $\nabla^2(f(\rho)\hat e_\rho) = \hat e_\rho\left(f'' + \frac{f'}{\rho} - \frac{f}{\rho^2}\right)$

(ii) $\nabla\cdot(f(\rho)\hat e_\rho) = \nabla\cdot(f(\rho)\hat e_\rho + 0e_\theta + 0e_\psi)$

$$= \frac{1}{h_\rho h_\theta h_\psi}\left\{\frac{\partial}{\partial\rho}(h_\theta h_\psi f(\rho))\right\}$$

$$= \frac{1}{\rho^2\sin\theta}\left[\frac{\partial}{\partial\rho}(\rho^2\sin\theta\, f(\rho))\right]$$

$$= \frac{1}{\rho^2}\left[\frac{\partial}{\partial\rho}(\rho^2 f(\rho))\right] = f'(\rho) + \frac{2}{\rho}f(\rho)$$

$$\nabla \cdot (\nabla \cdot f(\rho)\hat{e}_\rho) = \nabla \left(f'(\rho) + \frac{2}{\rho} f(\rho) \right)$$

$$= \hat{e}_\rho \frac{\partial}{\partial \rho}\left(f' + \frac{2}{\rho} f\right) + \frac{\hat{e}_\theta}{\rho} \frac{\partial}{\partial \theta}\left(f' + \frac{2}{\rho} f\right) + \frac{\hat{e}}{\rho \sin\theta}\left(f' + \frac{2}{\rho} f\right)$$

$$\Rightarrow \hat{e}_\rho \left[f''(\rho) + 2f'\frac{(\rho)}{\rho} - \frac{2f}{\rho^2} \right] \cdots\cdots \text{I}$$

(b) $\nabla \times f(\rho)\hat{e}_\rho$

$$= \begin{vmatrix} \frac{\hat{e}_\rho}{\rho^2 \sin\theta} & \frac{\hat{e}_\theta}{\rho \sin\theta} & \frac{\hat{e}_\psi}{\rho} \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ f(\rho) & 0 & 0 \end{vmatrix}$$

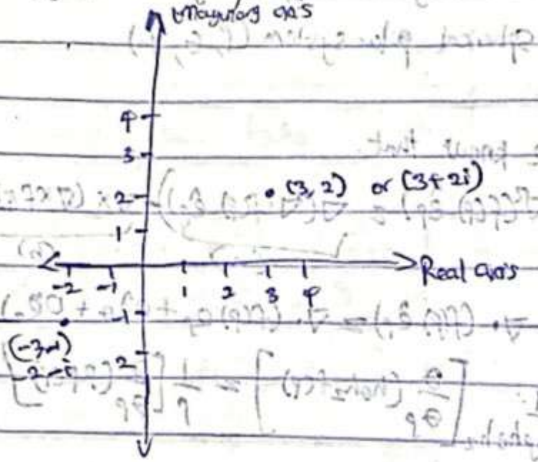
$$= \vec{0} \cdots\cdots \text{II}$$

From eqn (I) and (II)

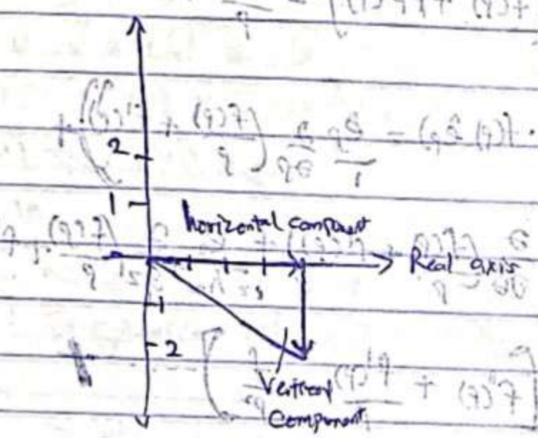
$$\nabla^2 (f(\rho)\hat{e}_\rho) = \hat{e}_\rho \left(f'' + \frac{2f}{\rho} - \frac{2f}{\rho^2} \right)$$

Complex Number Class Notes

Fig 1: Argand dig



Vector representation of $4-2i$



Addition and Substraction of Complex Number

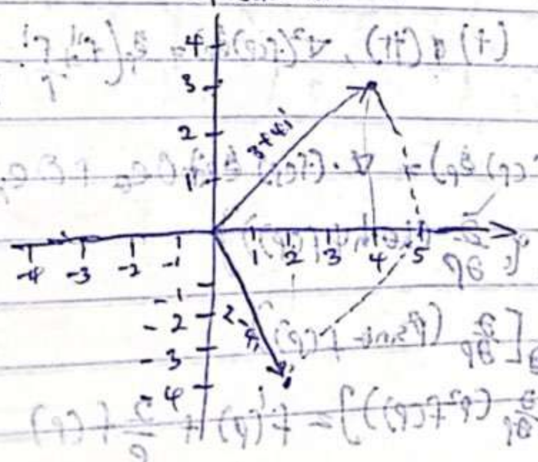
$(a+bi)+(c+di) = (a+c)+(b+d)i$

$(a+bi)-(c+di) = (a-c)+(b-d)i$

Example

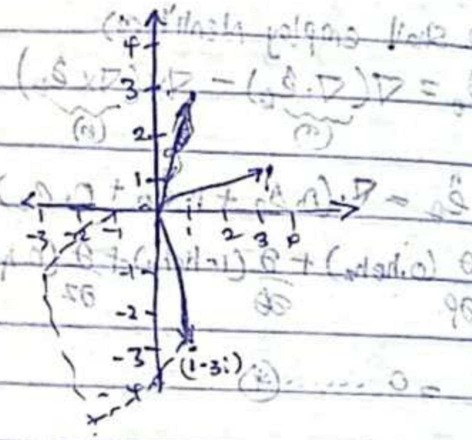
(1) $(2-4i)+(3+4i) = 5$

Vector representation



(2) $(1-3i)-(3+i) = -2-4i$

Vector representation



24th April, 2024

Multiplication of Complex Numbers

$(a+bi)(c+di) = a(c+di) + bi(c+di)$

$= ac + adi + bci + (bd)i^2$

$= ac + adi + bci - bd$

$= (ac-bd) + (ad+bc)i$

real part — imaginary part

Powers of the number (i)

$i^n = (i^{-1})^{(-n)} = \left(\frac{1}{i}\right)^{-n}$

$\left(\frac{1}{i}\times\frac{i}{i}\right)^{-n} \Rightarrow \left(\frac{i}{-1}\right)^{-n} = (-i)^{-n}$

Example 6

$x^3 - 3xy$

Complex Matrices

$$A = \begin{bmatrix} i & 1+i \\ 2-3i & 4 \end{bmatrix}$$

$$B = \begin{bmatrix} 2i & 0 \\ 1 & 1+2i \end{bmatrix}$$

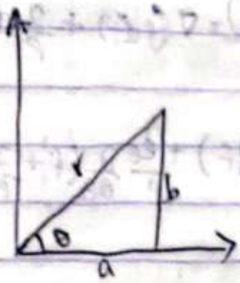
$$A+B = \begin{bmatrix} 3i & 1+i \\ 2-2i & 5+2i \end{bmatrix}$$

__Finding Determinant of Complex Number__

__Exercise__

Start from (d)

24th April, 2024



$\sin\theta = \frac{b}{r} \Rightarrow b = r\sin\theta$

$\cos\theta = \frac{a}{r} \Rightarrow a = r\cos\theta$

$a+bi = r\cos\theta + (r\sin\theta)i$

$\tan\theta = b/a$

To determine the polar argument, we study the following cases:

(a) If $x \neq 0$ from $\tan\theta = y/x$, we deduce that $\theta = \tan^{-1}\frac{y}{x} + k\pi$

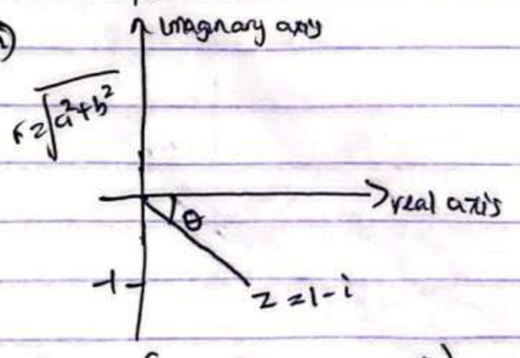
$$k = \begin{cases} 0, & \text{for } x > 0 \text{ and } y \geq 0 \\ 1, & \text{for } x < 0 \text{ and any } y \\ 2, & \text{for } x > 0 \text{ and } y < 0 \end{cases}$$

(b) If $x = 0$, and $y \neq 0$, then,

$$\theta = \begin{cases} \pi/2, & \text{for } y > 0 \\ 3\pi/2, & \text{for } y < 0 \end{cases}$$

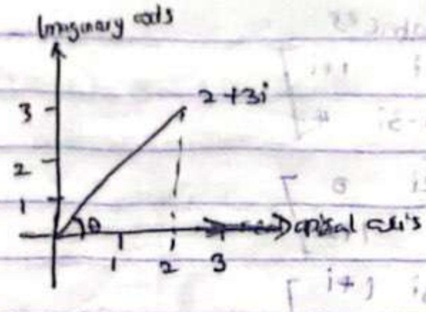
Example 13

(a)



$z = \sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right)$

(b)



$z = \sqrt{13}\left(\cos(0.98) + i\sin(0.98)\right)$

(c)

Converting from Polar to Standard form

$z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$, $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

$z_1 \cdot z_2 = r_1 r_2 [\cos\theta_1(\cos\theta_2 + i\sin\theta_2)] + i\sin\theta_1 (\cos\theta_2 + i\sin\theta_2)$

$z_1 z_2 = r_1 r_2 [\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$

From this result, we see that $|z_1 z_2|$ can be expressed as $|z_1| \times |z_2|$

and $\arg(z_1 z_2) = \arg z_1 + \arg z_2$

$$\frac{z_1}{z_2} = \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)}$$

$$= \frac{r_1}{r_2}\left[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\right]$$

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$$

$$\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2$$

De Moivre's Theorem

$z = r(\cos\theta + i\sin\theta)$

$z^2 = z \cdot z = r^2(\cos\theta + i\sin\theta)(\cos\theta + i\sin\theta)$

$= r^2(\cos 2\theta + i\sin 2\theta)$

$z^3 = z^2 \cdot z = r^3(\sin 2\theta + i\sin 2\theta)(\cos\theta + \sin\theta)$

$= r^3(\cos 3\theta + i\sin 3\theta)$

$z^n = r^n(\cos n\theta + i\sin n\theta)$

30th April, 2024

Finding nth root of a Complex Numbe

$\sqrt[n]{r}\left[\cos\left(\frac{\theta + 2\pi k}{n}\right) + i\sin\left(\frac{\theta + 2\pi k}{n}\right)\right]$

Expression of powers

$z = \cos\theta + i\sin\theta$

$z^n = \cos n\theta + i\sin n\theta$

$\frac{1}{z} = \frac{1}{\cos\theta + i\sin\theta} \times \frac{\cos\theta - i\sin\theta}{\cos\theta - i\sin\theta}$

$= \cos\theta - i\sin\theta$

$\frac{1}{z^n} = \frac{1}{\cos n\theta + i\sin n\theta} = \cos n\theta - i\sin n\theta$

Hence

$z + \frac{1}{z} = 2\cos\theta, \quad z - \frac{1}{z} = 2i\sin\theta$

$e^{i\theta} = \cos\theta + i\sin\theta$

$e^{-i\theta} = \cos\theta - i\sin\theta$

$\cos\theta = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right)$

If $z = x + iy$, find $\frac{z+1}{z+i} = 1$

$\Rightarrow \frac{x + iy + 1}{x + iy + i} = \frac{(x+1) + iy}{x + (y+1)i}$

rationalize

$\frac{(x+1) + iy}{x + (y+1)i} \times \frac{x - (y+1)i}{x - (y+1)i}$

$3z^2 - (2 + 11i)z + 3 - 5i = 0$

Using quadratic formular

$\frac{(2+11)i \pm \sqrt{-(2+11i)^2 - 4 \times 3(3 - 5i)}}{6}$

[illegible]

Complex Number Slides

MTH 202: Complex Numbers

April 21, 2024

0.1 Complex Numbers

In algebra it is often necessary to solve quadratic equations such as

$$x^2 - 3x + 2 = 0.$$

The general quadratic equation is $ax^2 + bx + c = 0$, and its solutions are given by the Quadratic Formula,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

where the quantity under the radical, $b^2 - 4ac$, is called the **discriminant.** If $b^2 - 4ac \geq 0$, then the solutions are ordinary real numbers. But what can we conclude about the solutions of a quadratic equation whose discriminant is negative? For example, the equation $x^2 + 4 = 0$ has a discriminant of $b^2 - 4ac = -16$. In algebra, it is clear that there is no real number whose square is -16. By writing

$$\sqrt{-16} = \sqrt{16(-1)} = \sqrt{16}\sqrt{-1} = 4\sqrt{-1},$$

we can see that the essence of the problem is that there is no real number whose square is -1. To solve the problem, mathematicians invented the **imaginary unit** i, which has the property $i^2 = -1$. In terms of this imaginary unit, we can write

$$\sqrt{-16} = 4\sqrt{-1} = 4i.$$

Definition 1. *The number i is called the* **imaginary unit** *and is defined as*

$$i = \sqrt{-1}.$$

Remark: When working with products involving square roots of negative numbers, be sure to convert to a multiple of i before multiplying. For instance, consider the following operations:

$$\sqrt{-1}\sqrt{-1} = i \cdot i = i^2 = -1 \qquad \textit{Correct}$$

$$\sqrt{-1}\sqrt{-1} = \sqrt{(-1)(-1)} = \sqrt{1} = 1. \qquad \textit{Incorrect}$$

Definition 2. *If a and b are real numbers, then the number*

$$a + bi$$

is a **complex number,** *where a is the* **real part** *and bi is the* **imaginary part** *of the number. The form $a + bi$ is the* **standard form** *of a complex number.*

A complex number is uniquely determined by its real and imaginary parts. So, we can say that two complex numbers are equal if and only if their real and imaginary parts are equal. That is, if $a + bi$ and $c + di$ are two complex numbers written in standard form, then

$$a + bi = c + di,$$

if and only if $a = c$ and $b = d$.

0.1.1 The Complex Plane

Because a complex number is uniquely determined by its real and imaginary parts, it is natural to associate the number $a + bi$ with the ordered pair (a, b). With this association, we can graphically represent complex numbers as points in a coordinate plane called the **complex plane.** This plane is an adaptation of the rectangular coordinate plane. Specifically, the horizontal axis is the **real axis** and the vertical axis is the **imaginary axis.** For instance, Figure 1 shows the graph of two complex numbers, $3 + 2i$ and $-2 - i$. The number $3 + 2i$ is associated with the point $(3, 2)$ and the number $-2 - i$ is associated with the point $(-2, -1)$.

Another way to represent the complex number $a+bi$ is as a vector whose horizontal component is a and whose vertical component is b (Figure 2.)

0.1.2 Addition and Scalar Multiplication of Complex Numbers

Because a complex number consists of a real part added to a multiple of i, the operations of addition and multiplication are defined in a manner consistent with the rules for operating with real numbers. For instance, to add (or subtract) two complex numbers, add (or subtract) the real and imaginary parts separately.

Definition 3. Addition and Subtraction of Complex Numbers

The **sum** and **difference** of $a+bi$ and $c+di$ are respectively defined as follows:

$$(a+bi)+(c+di)=(a+c)+(b+d)i$$

$$(a+bi)-(c+di)=(a-c)+(b-d)i.$$

Example 1. Adding and Subtracting Complex Numbers

$$(2-4i)+(3+4i)=(2+3)+(-4+4)i=5$$

$$(1-3i)-(3+i)=(1-3)+(-3-1)i=-2-4i.$$

Using the vector representation of complex numbers, we can add or subtract two complex numbers geometrically using the parallelogram rule for vector addition, as shown in Figure 3.

Many of the properties of addition of real numbers are valid for complex numbers as well. For instance, addition of complex numbers is both associative and commutative. Moreover, to find the sum of three or more complex numbers, extend the definition of addition in the natural way. For example,

$$(2+i)+(3-2i)+(-2+4i) = (2+3-2)+(1-2+4)i = 3+3i.$$

Definition 4. Definition of Scalar Multiplication
If c is a real number and $a+bi$ is a complex number, then the **scalar multiple** *of c and $a+bi$ s defined as*

$$c(a+bi) = ca+cbi.$$

Geometrically, multiplication of a complex number by a real scalar corresponds to the multiplication of a vector by a scalar, as shown in Figure 4.

Example 2. Operations with Complex Numbers

$$3(2+7i)+4(8-i) = 6+21i+32-4i = 38+17i$$
$$-4(1+i)+2(3-i)-3(1-4i) = -4-4i+6-2i-3+12i = -1+6i.$$

With addition and scalar multiplication, the set of complex numbers forms a vector space of dimension 2 (where the scalars are the real numbers).

0.1.3 Multiplication of Complex Numbers

The operations of addition, subtraction, and multiplication by a real number have exact counterparts with the corresponding vector operations. By contrast, there is no direct counterpart for the multiplication of two complex numbers.

Definition 5. *The* **product** *of the complex numbers $a + bi$ and $c + di$ is defined as*

$$(a + bi)(c + di) = (ac - bd) + (ad + bc)i.$$

Example 3.

$$(-2)(1 - 3i) = -2 + 6i$$

$$(2 - i)(4 + 3i) = 8 + 6i - 4i - 3i^2 = 8 + 6i - 4i - 3(-1) = 8 + 3 + 6i - 4i = 11 + 2i.$$

Example 4. Complex Zeros of a Polynomial:
Use the Quadratic Formula to find the zeros of the polynomial

$$p(x) = x^2 - 6x + 13$$

and verify that $p(x) = 0$ for each zero.

Solution: Using the Quadratic Formula, we have

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{6 \pm \sqrt{36 - 52}}{2}$$

$$= \frac{6 \pm \sqrt{-16}}{2} = \frac{6 \pm 4i}{2} = 3 \pm 2i.$$

Substituting these values of x into the polynomial $p(x)$, we have

$$p(3 + 2i) = (3 + 2i)^2 - 6(3 + 2i) + 13 = 9 + 12i - 4 - 18 - 12i + 13 = 0$$

and

$$p(3 - 2i) = (3 - 2i)^2 - 6(3 - 2i) + 13 = 9 - 12i - 4 - 18 + 12i + 13 = 0.$$

In Example 4, the two complex numbers $3+2i$ and $3-2i$ are **complex conjugates** of each other (together they form a **conjugate pair**). A well-known result from algebra states that the complex zeros of a polynomial with real coefficients must occur in conjugate pairs.

0.1.4 Powers of the Number i

The formulas for the powers of a complex number with integer exponents are preserved for the algebraic form $z = x + iy$. Setting $z = i$, we obtain

$$i^0 = 1; \quad i^1 = i; \quad i^2 = -1; \quad i^3 = i^2 \cdot i = -i;$$

$$i^4 = i^3 \cdot i = 1; \quad i^5 = i^4 \cdot i = i; \quad i^6 = i^5 \cdot i = -1; \quad i^7 = i^6 \cdot i = -i.$$

It can be proved by induction that for every positive integer n,

$$i^{4n} = 1; \quad i^{4n+1} = i; \quad i^{4n+2} = -1; \quad i^{4n+3} = -i.$$

Hence, $i^n \in \{-1, 1, -i, i\}$ for all integers $n \geq 0$. If n is a negative integer, we have

$$i^n = (i^{-1})^{-n} = \left(\frac{1}{i}\right)^{-n} = (-i)^{-n}.$$

Example 5. *We have*

$$i^{105} + i^{23} + i^{20} - i^{34} = i^{4 \cdot 26+1} + i^{4 \cdot 5+3} + i^{4 \cdot 5} - i^{4 \cdot 8+2} = i - i + 1 + 1 = 2.$$

Example 6. *Solve the equation*

$$z^3 = 18 + 26i, \text{ where } z = x + iy \text{ and } x,\ y \text{ are integers.}$$

Solution:

$$(x + yi)^3 = (x + iy)^2(x + iy) = (x^2 - y^2 + 2xiy)(x + iy)$$

$$= (x^3 - 3xy^2) + (3x^2y - y^3)i = 18 + 26i.$$

Using the definition of equality of complex numbers, we obtain

$$x^3 - 3xy^2 = 18,$$

$$3x^2y - y^3 = 26.$$

Setting $y = tx$ in the equality $18(3x^2y - y^3) = 26(x^3 - 3xy^2)$, let us observe that $x \neq 0$ and $y \neq 0$ implies $18(3t - t^3) = 26(1 - 3t^2)$, which is equivalent to $(3t-1)(3t^2 - 12t - 13) = 0$.

The only rational solution of this equation is $t = \frac{1}{3}$; hence,

$$x = 3, \quad y = 1, \quad \text{and} \quad z = 3 + i.$$

0.1.5 Complex Matrices

Definition 6. *A matrix whose entries are complex numbers is called a* **complex matrix.**

Example 7. Operations with Complex Matrices:
Let A and B be the complex matrices below

$$A = \begin{bmatrix} i & 1+i \\ 2-3i & 4 \end{bmatrix}$$

and

$$B = \begin{bmatrix} 2i & 0 \\ i & 1+2i \end{bmatrix}$$

and determine each of the following: (a) 3A (b) (2 - i)B (c) A + B (d) BA

Solution:
(a)

$$3A = 3\begin{bmatrix} i & 1+i \\ 2-3i & 4 \end{bmatrix} = \begin{bmatrix} 3i & 3+3i \\ 6-9i & 12 \end{bmatrix}$$

(b)

$$(2-i)B = (2-i)\begin{bmatrix} 2i & 0 \\ i & 1+2i \end{bmatrix} = \begin{bmatrix} 2+4i & 0 \\ 1+2i & 4+3i \end{bmatrix}$$

(c)

$$A+B = \begin{bmatrix} i & 1+i \\ 2-3i & 4 \end{bmatrix} + \begin{bmatrix} 2i & 0 \\ i & 1+2i \end{bmatrix} = \begin{bmatrix} 3i & 1+i \\ 2-2i & 5+2i \end{bmatrix}$$

(d)

$$BA = \begin{bmatrix} 2i & 0 \\ i & 1+2i \end{bmatrix}\begin{bmatrix} i & 1+i \\ 2-3i & 4 \end{bmatrix} = \begin{bmatrix} -2 & -2+2i \\ 7+i & 3+9i \end{bmatrix}$$

Example 8. Finding the Determinant of a Complex Matrix
Find the determinant of the matrix

$$A = \begin{bmatrix} 2-4i & 2 \\ 3 & 5-3i \end{bmatrix}.$$

Solution:

$$det(A) = \begin{vmatrix} 2-4i & 2 \\ 3 & 5-3i \end{vmatrix} = (2-4i)(5-3i)-(2)(3) = 10-20i-6i-12-6 = -8-26i.$$

EXERCISES

(a) Determine the values of the expressions

i. $\sqrt{-2}\sqrt{-3}, \quad \sqrt{-4}\sqrt{-4}, \quad \sqrt{8}\sqrt{-8}$

ii. $i^3, \quad i^4, \quad (-i)^7$

(b) Plot the following complex numbers.

i. $z = 6 - 2i, \quad z = 1 + 5i.$

ii. $z = 3i, \quad z = 7.$

(c) Determine x such that the complex numbers in each pair are equal.

i. $x + 3i, \ 6 + 3i$

ii. $(2x - 8) + (x - 1)i, \ 2 + 4i$

iii. $(-x + 4) + (x + 1)i, \ x + 3i$

(d) Solve the following equations:

i. $z \cdot (1, 2) = (-1, 3);$

ii. $(1, 1) \cdot z^2 = (-1, 7).$

(e) Compute the following:

i. $i^{2000} + i^{1999} + i^{201} + i^{82} + i^{47};$

ii. $i^{-5} + (-i)^{-7} + (-i)^{13} + i^{-100} + (-i)^{94}$

(f) Solve in $\mathbb{C}$ the following equations:

i. $z^2 = i$

ii. $z^2 = -i$

iii. $z^2 = \frac{1}{2} - i\frac{\sqrt{2}}{2}.$

(g) Let a, b, c be real numbers and $\omega = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$. Compute

$$(a + b\omega + c\omega^2)(a + b\omega^2 + c\omega).$$

(h) Perform the indicated matrix operation using the complex matrices A and B

$$A = \begin{bmatrix} 1+i & 1 \\ 2-2i & -3i \end{bmatrix}$$

and

$$B = \begin{bmatrix} 1-i & 3i \\ -3 & -i \end{bmatrix}$$

(i) $A+B$ (ii) $B-A$ (iii) $2iA$ (iv) $\frac{1}{4}iB$.

0.2 Conjugates and Division of Complex Numbers

0.2.1 Definition of the Conjugate of a Complex Number

Definition 7. *The* **conjugate** *of the complex number $z = a+bi$ is denoted by $\bar{z}$ and is given by*

$$\bar{z} = a - bi.$$

From this definition, we can see that the conjugate of a complex number is found by changing the sign of the imaginary part of the number. For example,

$$z = -2+3i \Longrightarrow \bar{z} = -2-3i, \quad z = 4-5i \Longrightarrow \bar{z} = 4+5i.$$

Remark: It should be noted that a number is its own complex conjugate if and only if the number is real.

Geometrically, two points in the complex plane are conjugates if and only if they are reflections about the real (horizontal) axis, as shown in Figure 5.

0.2.2 Properties of Complex Conjugates

For complex number $z = a + bi$, the following properties are true.

i. $z\bar{z} = a^2 + b^2$

ii. $z\bar{z} \geq 0$

iii. $z\bar{z} = 0$ if and only if $z = 0$

iv. $(\bar{\bar{z}}) = z$.

Example 9. *Find the product of $z = 1 - 2i$ and its complex conjugate.*

Solution: Because $\bar{z} = 1 + 2i$, we have

$$z\bar{z} = (1 - 2i)(1 + 2i) = 1^2 + 2^2 = 1 + 4 = 5.$$

0.2.3 The Modulus of a Complex Number

Because a complex number can be represented by a vector in the complex plane, it makes sense to talk about the length of a complex number. This length is called the modulus of the complex number.

Definition 8. *The* **modulus** *of the complex number $z = a + bi$ is denoted by $|z|$ and is given by*

$$|z| = \sqrt{a^2 + b^2}.$$

Example 10. Finding the Modulus of a Complex Number
For $z = 2 + 3i$ and $w = 6 - i$, determine the value of each modulus. (a) $|z|$ (b) $|w|$ (c) $|zw|$.

Solutions:
(a) $|z| = \sqrt{2^2 + 3^2} = \sqrt{13}$; (*b*) $|w| = \sqrt{6^2 + (-1)^2} = \sqrt{37}$;
(*c*) $zw = (2 + 3i)(6 - i) = 15 + 16i \Longrightarrow |zw| = \sqrt{15^2 + 16^2} = \sqrt{481}$.

Theorem 1. *For a complex number z,*

$$|z|^2 = z\bar{z}.$$

0.2.4 Division of Complex Number

Consider $z = a + bi$ and $w = c + di$ and assume that c and d are not both 0. If the quotient

$$\frac{z}{w} = x + iy$$

is to make sense, it has to be true that

$$z = w(x + iy) = (c + di)(x + iy) = (cx - dy) + (dx + cy)i.$$

But, because $z = a + bi$, we can form the linear system

$$cx - dy = a$$

$$dx + cy = b.$$

Solving this system of linear equations for x and y yields

$$x = \frac{ac + bd}{w\bar{w}}, \quad y = \frac{bc - ad}{w\bar{w}}.$$

Definition 9. *The* **quotient** *of the complex numbers $z = a + bi$ and $w = c + di$ is defined as*

$$\frac{z}{w} = \frac{a + bi}{c + di} = \frac{ac + bd}{c^2 + d^2} + \frac{bc - ad}{c^2 + d^2}i = \frac{1}{|w|^2}(z\bar{w}),$$

provided $c^2 + d^2 \neq 0$.

In practice, the quotient of two complex numbers can be found by multiplying the numerator and the denominator by the conjugate of the denominator, as follows:

$$\frac{a + bi}{c + di} = \frac{a + bi}{c + di}\left(\frac{c - di}{c - di}\right) = \frac{(a + bi)\,(c - di)}{(c + di)\,(c - di)} = \frac{(ac + bd) + (bc - ad)i}{c^2 + d^2}$$

$$= \frac{ac + bd}{c^2 + d^2} + \frac{bc - ad}{c^2 + d^2}i.$$

Example 11.

$$(a) \quad \frac{1}{1 + i} = \frac{1}{1 + i}\left(\frac{1 - i}{1 - i}\right) = \frac{1 - i}{1^2 - i^2} = \frac{1 - i}{2} = \frac{1}{2} - \frac{1}{2}i$$

$$(b) \quad \frac{2 - i}{3 + 4i} = \frac{2 - i}{3 + 4i}\left(\frac{3 - 4i}{3 - 4i}\right) = \frac{2 - 11i}{9 + 16} = \frac{2}{25} - \frac{11}{25}i.$$

Example 12. *Find the inverse of the matrix*

$$A = \begin{bmatrix} 2-i & -5+2i \\ 3-i & -6+2i \end{bmatrix}$$

and verify your solution by showing that $AA^{-1} = I_2$.

Solution: Using the formula for the inverse of a 2×2 matrix, we have

$$A^{-1} = \frac{1}{|A|} \begin{bmatrix} -6+2i & 5-2i \\ -3+i & 2-i \end{bmatrix}$$

Furthermore, because

$$|A| = (2-i)(-6+2i) - (5-2i)(-3+1) = 3-i,$$

we can write

$$A^{-1} = \frac{1}{3-i} \begin{bmatrix} -6+2i & 5-2i \\ -3+i & 2-i \end{bmatrix} = \frac{1}{3-i}\left(\frac{1}{3+i}\right) \begin{bmatrix} (-6+2i)(3+i) & (5-2i)(3+i) \\ (-3+i)(3+i) & (2-i)(3+i) \end{bmatrix}$$

$$= \frac{1}{10} \begin{bmatrix} -20 & 17-i \\ -10 & 7-i \end{bmatrix}.$$

To verify, we multiply A and A^{-1} as follows:

$$AA^{-1} = \begin{bmatrix} 2-i & -5+2i \\ 3-i & -6+2i \end{bmatrix} \frac{1}{10} \begin{bmatrix} -20 & 17-i \\ -10 & 7-i \end{bmatrix} = \frac{1}{10} \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Theorem 2. Properties of Complex Conjugates
For the complex numbers z and w, the following properties are true.

i. $\overline{z+w} = \bar{z} + \bar{w}$

ii. $\overline{z-w} = \bar{z} - \bar{w}$

iii. $\overline{zw} = \bar{z}\bar{w}$

iv. $\overline{\left(\frac{z}{w}\right)} = \frac{\bar{z}}{\bar{w}}$.

EXERCISES

a. Find the indicated modulus, where $z = 2 + i$, $w = -3 + 2i$, and $v = -5i$. (i) $|z|$ (ii) $|z^2|$ (iii) $|zw|$ (iv) $|v|$ (v) $|zv^2|$.

b. Verify that $\left(\frac{1+i}{\sqrt{2}}\right)^2 = i$.

c. Determine all values of the complex number z for which A is singular. (Hint: Set $det(A) = 0$ and solve for z)

$$A = \begin{bmatrix} 5 & z \\ 3i & 2-i \end{bmatrix}, \quad A = \begin{bmatrix} 2 & 2i & 1+i \\ 1-i & -1+i & z \\ 1 & 0 & 0 \end{bmatrix}$$

(d) Solve the following equations:

i. $|z| - 2z = 3 - 4i$;

ii. $|z| + z = 3 + 4i$;

iii. $z^3 = 2 + 11i$, where $z = x + iy$ and $x, y \in \mathbb{Z}$

iv. $(1 + i)z^2 + 2 + 11i = 0$.

(e) Find the real numbers x and y in each of the following cases:

i. $(1 - 2i)x + (1 + 2i)y = 1 + i$;

ii. $\frac{x-3}{3+i} + \frac{y-3}{3-i} = i$;

iii. $(4 - 3i)x^2 + (3 + 2i)xy = 4y^2 - \frac{1}{2}x^2 + (3xy - 2y^2)i$.

(f) Let $z_1 = 1 + i$ and $z_2 = -1 - i$. Find $z_3 \in \mathbb{C}$ such that triangle z_1, z_2, z_3 is equilateral.

(g) Find all complex numbers z such that

$$4z^2 + 8|z|^2 = 8.$$

0.3 Complex Numbers in Trigonometric Form

0.3.1 Polar Form and DeMoivre's Theorem

To work effectively with powers and roots of complex numbers, it is helpful to use a polar representation for complex numbers, as shown in Figure 6.

Specifically, if $a + bi$ is a nonzero complex number, then let θ be the angle from the positive x-axis to the radial line passing through the point (a, b) and let r be the modulus of $a + bi$. So,

$$a = r\cos\theta, \quad b = r\sin\theta, \ \textit{and} \ r = \sqrt{a^2 + b^2}$$

and we have

$$a + bi = (r\cos\theta) + (r\sin\theta)i,$$

from which the **polar form** of a complex number is obtained.

Definition 10. *The* **polar form** *of the nonzero complex number* $z = a + bi$ *is given by*

$$z = r(\cos\theta + i\sin\theta),$$

where $a = r\cos\theta$, $b = r\sin\theta$, $r = \sqrt{a^2 + b^2}$, *and* $\tan\theta = b/a$. *The number* r *is the* **modulus** *of* z *and* θ *is the* **argument** *of* z.

Remark: The polar form of $z = 0$ is expressed as $z = 0(\cos\theta + i\sin\theta)$, where θ is any angle.

Because there are infinitely many choices for the argument, the polar form of a complex number is not unique. Normally, the values of θ that lie between $-\pi$ and π are used, although on occasion it is convenient to use other values. The value of θ that satisfies the inequality

$$-\pi < \theta < \pi$$

is called the **principal argument** and is denoted by $Arg(z)$. Two nonzero complex numbers in polar form are equal if and only if they have the same modulus and the same principal argument.

Example 13. *Find the polar form of each of the complex numbers. (Use the principal argument.) (a) $z = 1 - i$ (b) $z = 2 + 3i$ (c) $z = i$.*

Solution:
(a) Because $a = 1$ and $b = -1$, then $r^2 = 1^2 + (-1)^2 = 2 \Longrightarrow r = \sqrt{2}$.
From $a = r\cos\theta$ and $b = r\sin\theta$, we have

$$\cos\theta = \frac{a}{r} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \quad \textit{and} \quad \sin\theta = \frac{b}{r} = -\frac{1}{\sqrt{2}} = -\frac{\sqrt{2}}{2}.$$

So,

$$\theta = -\pi/4 \quad \textit{and} \quad z = \sqrt{2}\Big[\cos\Big(-\frac{\pi}{4}\Big) + i\sin\Big(-\frac{\pi}{4}\Big)\Big].$$

(b) Because $a = 2$ and $b = 3$, then $r^2 = 2^2 + 3^2 = 2 \Longrightarrow r = \sqrt{13}$. So,

$$\cos\theta = \frac{a}{r} = \frac{2}{\sqrt{13}} \quad \textit{and} \quad \sin\theta = \frac{b}{r} = \frac{3}{\sqrt{13}}$$

and it follows that $\theta \approx 0.98$. So, the polar form is

$$z \approx \sqrt{13}\Big[\cos(0.98) + i\sin(0.98)\Big].$$

(c) Because $a = 0$ and $b = 1$, it follows that $r = 1$ *and* $\theta = \pi/2$, so

$$z = 1\Big(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\Big).$$

The polar forms derived in parts (a), (b), and (c) are depicted graphically in Figure 7.

Example 14. Converting from Polar to Standard Form
Express the complex number in standard form.

$$z = 8\Big[\cos\Big(-\frac{\pi}{3}\Big) + i\sin\Big(-\frac{\pi}{3}\Big)\Big].$$

Solution: Because $\cos\big(-\frac{\pi}{3}\big) = \frac{1}{2}$ and $\sin\big(-\frac{\pi}{3}\big) = -\frac{\sqrt{3}}{2}$, we can obtain the standard form

$$z = 8\Big[\cos\Big(-\frac{\pi}{3}\Big) + i\sin\Big(-\frac{\pi}{3}\Big)\Big] = 8\Big[\frac{1}{2} - i\frac{\sqrt{3}}{2}\Big] = 4 - 4\sqrt{3}i.$$

Suppose we have two complex numbers in polar form

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1) \quad \textit{and} \quad z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$$

Then the product of z_1 and z_2 is expressed as

$$z_1 z_2 = r_1 r_2(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)$$

$$= r_1 r_2\Big[\Big(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2\Big) + i\Big(\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2\Big)\Big].$$

Using the trigonometric identities

$$\cos(\theta_1 + \theta_2) = \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2$$

and

$$\sin(\theta_1 + \theta_2) = \sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2,$$

we have

$$z_1 z_2 = r_1 r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)].$$

Theorem 3. Product and Quotient of Two Complex Numbers:
Given two complex numbers in polar form

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1) \quad \textit{and} \quad z_2 = r_2(\cos\theta_2 + i\sin\theta_2),$$

the product and quotient of the numbers are as follows.

$$z_1 z_2 = r_1 r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)] \qquad \textit{Product}$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)], \quad z_2 \neq 0 \qquad \textit{Quotient.}$$

This theorem says that
(i) to multiply two complex numbers in polar form, multiply moduli and add arguments;
(ii) to divide two complex numbers, divide moduli and subtract arguments.

Example 15. *Find $z_1 z_2$ and $\frac{z_1}{z_2}$ for the complex numbers*

$$z_1 = 5\left[\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right] \quad \textit{and} \quad z_2 = \frac{1}{3}\left[\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right].$$

Solution: Because we have the polar forms of z_1 and z_2, we can apply Theorem 3 as follows:

$$z_1 z_2 = (5)\left(\frac{1}{3}\right)\left[\cos\left(\frac{\pi}{4} + \frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{4} + \frac{\pi}{6}\right)\right] = \frac{5}{3}\left[\cos\frac{5\pi}{12} + i\sin\frac{5\pi}{12}\right].$$

$$\frac{z_1}{z_2} = \left(\frac{5}{1/3}\right)\left[\cos\left(\frac{\pi}{4} - \frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{4} - \frac{\pi}{6}\right)\right] = 15\left[\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right].$$

0.4 DeMoivre's Theorem

Repeated use of multiplication in the polar form yields

$$z = r(\cos\theta + i\sin\theta)$$

$$z^2 = r(\cos\theta + i\sin\theta)r(\cos\theta + i\sin\theta) = r^2(\cos 2\theta + i\sin 2\theta)$$

$$z^2 = r(\cos\theta + i\sin\theta)r^2(\cos 2\theta + i\sin 2\theta) = r^3(\cos 3\theta + i\sin 3\theta).$$

Similarly,

$$z^4 = r^4(\cos 4\theta + i\sin 4\theta)$$

$$z^5 = r^5(\cos 5\theta + i\sin 5\theta).$$

Theorem 4. *If $z = r(\cos\theta + i\sin\theta)$ and n is any positive integer, then*

$$z^n = r^n(\cos n\theta + i\sin n\theta).$$

Example 16. Raising a Complex Number to an Integer Power
Find $(-1 + \sqrt{3}i)^{12}$ and write the result in standard form.

Solution: First convert to polar form. For $-1+\sqrt{3}i$

$$r = \sqrt{(-1)^2 + (\sqrt{3})^2} = 2 \ \ and \ \ \tan\theta = \frac{\sqrt{3}}{-1} = -\sqrt{3}$$

which implies that $\theta = 2\pi/3$. So,

$$-1+\sqrt{3}i = 2\left[\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right]$$

By DeMoivre's Theorem,

$$\begin{aligned}
(-1+\sqrt{3}i)^{12} &= \left[2\left(\cos\left(\frac{12(2\pi)}{3}\right) + i\sin\left(\frac{12(2\pi)}{3}\right)\right)\right]^{12} \\
&= 2^{12}\left(\cos\left(\frac{12(2\pi)}{3}\right) + i\sin\left(\frac{12(2\pi)}{3}\right)\right) \\
&= 4096(\cos 8\pi + i\sin 8\pi) \\
&= 4096(1+i(0)) = 4096.
\end{aligned}$$

Definition 11. nth Root of a Complex Number
The complex number $w = a+bi$ is an **nth root** *of the complex number z if*

$$z = w^n = (a+bi)^n.$$

DeMoivres Theorem is useful in determining roots of complex numbers. Let w be an nth root of z, where

$$w = s(\cos\beta + i\sin\beta) \quad z = r(\cos\theta + i\sin\theta).$$

Then, by DeMoivre's Theorem, we have $w^n = s^n(\cos n\beta + i\sin n\beta)$ and because $w^n = z$, it follows that

$$s^n(\cos n\beta + i\sin n\beta) = r(\cos\theta + i\sin\theta).$$

Now, because the right and left sides of this equation represent equal complex numbers, we can equate moduli to obtain $s^n = r$, which implies that $s = \sqrt[n]{r}$ and equate principal arguments to conclude that θ and $n\beta$ must differ by a multiple of 2π.

Note that r is a positive real number and so $s = \sqrt[n]{r}$ is also a positive real number. Consequently, for some integer k, $n\beta = \theta + 2\pi k$, which implies that

$$\beta = \frac{\theta + 2\pi k}{n}.$$

Finally, substituting this value of β into the polar form of w produces the result stated in the next theorem.

Theorem 5. The nth Root of a Complex Number:
For any positive integer n, the complex number $z = r(\cos\theta + i\sin\theta)$ has exactly n distinct roots. These n roots are given by

$$\sqrt[n]{r}\left[\cos\left(\frac{\theta + 2\pi k}{n}\right) + i\sin\left(\frac{\theta + 2\pi k}{n}\right)\right],$$

where $k = 0, 1, 2 \ldots, n-1$.

Remark: Note that when k exceeds $n-1$, the roots begin to repeat. For instance, if $k = n$, the angle is

$$\frac{\theta + 2\pi n}{n} = \frac{\theta}{n} + 2\pi$$

which yields the same values for the sine and cosine as $k = 0$.
Note: When Theorem 5 is applied to the real number 1, the *nth* roots have a special name- the **nth roots of unity.**

Example 17. Finding the nth Roots of a Complex Number:
Determine the fourth roots of i.

Solution: In polar form, we can write i as

$$i = 1\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$$

so that $r = 1$ and $\theta = \pi/2$. Then, by applying Theorem 5, we have

$$\begin{aligned} i^{1/4} &= \sqrt[4]{1}\left[\cos\left(\frac{\pi/2}{4} + \frac{2k\pi}{4}\right) + i\sin\left(\frac{\pi/2}{4} + \frac{2k\pi}{4}\right)\right] \\ &= \cos\left(\frac{\pi}{8} + \frac{k\pi}{2}\right) + i\sin\left(\frac{\pi}{8} + \frac{k\pi}{2}\right). \end{aligned}$$

Setting $k = 0, 1, 2,$ and 3, we obtain the four roots

$$z_1 = \cos\frac{\pi}{8} + i\sin\frac{\pi}{8},$$

$$z_2 = \cos\frac{5\pi}{8} + i\sin\frac{5\pi}{8},$$

$$z_3 = \cos\frac{9\pi}{8} + i\sin\frac{9\pi}{8},$$

$$z_4 = \cos\frac{13\pi}{8} + i\sin\frac{13\pi}{8}.$$

EXERCISES

1. Compute
$$z = \frac{(1-i)^{10}(\sqrt{3}+i)^5}{(-1-i\sqrt{3})^{10}}$$

2. Compute
$$(1+i)^{1000}$$

3. Find the polar representations for the following complex numbers:
$$(i.)\ z_1 = 6+6i\sqrt{3};\ \ (ii.)\ z_2 = 9-9i\sqrt{3};\ \ (iii)\ z_3 = -\frac{1}{4}+i\frac{\sqrt{3}}{4};\ \ z_4 = -\frac{1}{2}-i\frac{\sqrt{3}}{2}.$$

4. Given that z is a complex number such that $z + \frac{1}{z} = 2\cos 3^\circ$, find the least integer that is greater than $z^{2000} + \frac{1}{z^{2000}}$.

e. Compute $z^n + \frac{1}{z^n}$, if $z + \frac{1}{z} = \sqrt{3}$.

5. Find the square roots of the following complex numbers:
$$(i.)\ z = 1+i;\ \ (ii.)\ z = 7-24i;\ \ (iii)\ z = -2(1+i\sqrt{3});\ (iv)\ z = \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}.$$

6. Find the cube roots of the following complex numbers:
$$(i.)\ z = -i;\ \ (ii.)\ z = 18+26i;\ \ (iii)\ z = -27;\ \ (iv)\ z = \frac{1}{2} - i\frac{\sqrt{3}}{2}.$$

7. Find the fourth roots of the following complex numbers:

$$(i.)\ z = 2 - i\sqrt{12};\ \ (ii.)\ z = \sqrt{3} + i;\ \ (iii)\ z = -7 + 24i;\ \ (iv)\ z = -2i.$$

8. Find all the solutions to the equation and represent your solutions graphically.

$$(i.)\ x^4 - i = 0;\ \ (ii.)\ x^3 + 1 = 0;\ \ (iii.)\ x^3 + 64i = 0;\ \ (iv.)\ x^5 + 243 = 0.$$

9. Recall that the Maclaurin series for e^x, $\sin x$, and $\cos x$ are

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots.$$

   i. Substitute $x = e^{i\theta}$ in the series for e^x and show that

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

   ii. Show that any complex number $z = a + bi$ can be expressed in polar form as

$$z = re^{i\theta}.$$

   iii. Prove that if $z = re^{i\theta}$, then $\bar{z} = re^{-i\theta}$.

   iv. Prove the amazing formula

$$e^{i\pi} = -1.$$

Linear Algebra Class Notes

2nd May, 2024

Linear Vector Spaces

A field of Scalars $K = \mathbb{R} + \mathbb{C}$

(i) for any $u, v \in V$, $u+v \in V$

(ii) For any $u, v \in V$, $u+v = v+u \in V$

(iii) $\exists$ any $0 \in V \ni 0+v = v = v+0 \in V$
$\forall\ v \in V$

(iv) For any $u, v, w \in V$, $(u+v)+w$
$= u+(v+w)$

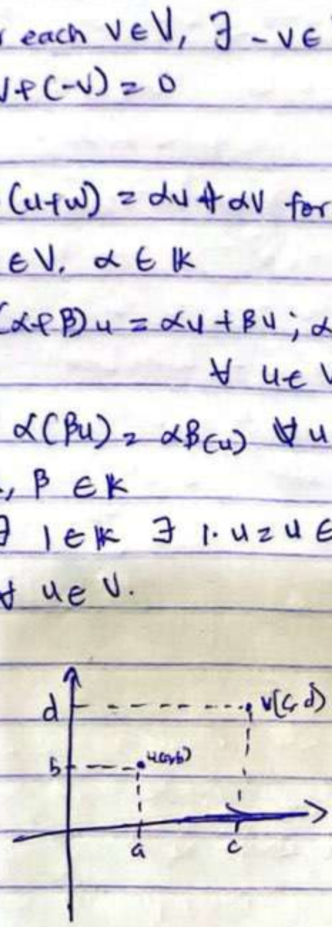
(v) For each $v \in V$, $\exists\ -v \in V$
$v+(-v) = 0$

(vi) $\alpha(u+w) = \alpha u + \alpha v$ for all
$u, v \in V$, $\alpha \in K$

(vii) $(\alpha+\beta)u = \alpha u + \beta u$; $\alpha, \beta \in K$
$\forall\ u \in V$

(viii) $\alpha(\beta u) = \alpha\beta(u)$ $\forall u \in V$,
$\alpha, \beta \in K$

(ix) $\exists\ 1 \in K \ni 1 \cdot u = u \in V$;
$\forall\ u \in V$.



$u+v = (a,b) + (c,d)$
$= (a+c, b,d)$

$V = \mathbb{R}^2$
$\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$
$u \in \mathbb{R}^2 \Rightarrow u = (a,b)$; $a, b \in \mathbb{R}$
$\therefore (a+c, b+d) \in \mathbb{R} \times \mathbb{R} = \mathbb{R}^2$

$(\mathbb{R}^2, +)$ $v+u = (c+a, d+b)$
$= (a+c, b+d)$
$= (a,b) + (c,d) = u + v$ Commutative

Examples

(1) $\mathbb{R}^n$, with operations addition, $+$ and scalar multiplication, $\cdot$; $(\mathbb{R}^n, +, \cdot)$ is a linear Vector space over a field of scalar; $\mathbb{R}$

i.e

$u, v \in \mathbb{R}^n$

$u = (a_1, a_2, a_3 \cdots, a_n)$
$a_i \in \mathbb{R}$

$v = (b_1, b_2, b_3 \cdots, b_n)$; $b_i \in \mathbb{R}$

$u+v = (a_1+b_1, a_2+b_2, a_3+b_3 \cdots, a_n+b_n)$

(i) For any $u, v \in \mathbb{R}^n$, $u+v \in \mathbb{R}^n$?
$u+v = (a_1+b_1, a_2+b_2, a_3+b_3 \cdots, a_n+b_n)$
$a_i+b_i \in \mathbb{R}$ for $i = 1, \cdots n$
$\therefore u+v \in \mathbb{R}^n$ for any $u, v \in \mathbb{R}^n$

(2) $\mathbb{P}_n(\mathbb{R})$ - Space of polynomials of degree less than or equal to n with coefficients in $\mathbb{R}$

i.e for any $q(x) \in \mathbb{P}_n(\mathbb{R})$;
$q(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$
$r(x) = b_0 + b_1 x + b_2 x^2 + \cdots + b_n x^n$
for $x \in \mathbb{R}$ and a_i } $i = 0, 1, \ldots n$

Let $q(x) \in \mathbb{P}_n(\mathbb{R})$ and $r(x) \in \mathbb{P}_n(\mathbb{R})$
$q(x) + r(x) = (a_0+b_0) + (a_1+b_1)x$

Linear Combination

$u_1, u_2, \ldots u_n \in V$

$u \in V$

$\alpha_1, \alpha_2, \alpha_3 \ldots, \alpha_n \in K$

$u = \alpha_1 u_1 + \alpha_2 u_2 + \ldots + \alpha_n u_n$

(If $u \in V$ and $v_i \in V$ where $i = 1, 2, 3 \ldots n$. u is a linear combination of v_i if; $u = \sum_{i}^{n} \alpha_i v_i$)

Let $A, B, C, D \in M_{2\times2}(\mathbb{R})$

(1) $D = \begin{bmatrix} 4 & 7 \\ 7 & 9 \end{bmatrix}$, $A \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

$B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 1 \\ 4 & 5 \end{bmatrix}$

Express D as a linear combination of A, B, C

D is a linear combination of A, B, C only if there exist scalars,

$\alpha, \beta, \gamma \in \mathbb{R}$, such that

$D = \alpha A + \beta B + \gamma C$

$\Rightarrow \begin{bmatrix} 4 & 7 \\ 7 & 9 \end{bmatrix} = \alpha \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + \beta \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \gamma \begin{bmatrix} 1 & 1 \\ 4 & 5 \end{bmatrix}$

$= \begin{bmatrix} \alpha+\beta+\gamma & \alpha+2\beta+\gamma \\ \alpha+3\beta+4\gamma & \alpha+4\beta+5\gamma \end{bmatrix}$

$\alpha + \beta + \gamma = 4 \ldots (i)$

$\Rightarrow \alpha + 2\beta + \gamma = 7 \ldots (ii)$

$\alpha + 3\beta + 4\gamma = 7 \ldots (iii)$

$\alpha + 4\beta + 5\gamma = 9 \ldots (iv)$

$\alpha = 2, \beta = 3, \gamma = -1$

$\therefore D = 2A + 3B - C$

3rd May, 2024

(2) If $w = (1, K, 4)$, $u = (1, 2, 3)$ and $v = (2, 3, 1) \in \mathbb{R}^3$

Find K so that w is a linear combination of u and v.

"If w is a linear combination of u and v, that means we can find a constant α and β such that w can be expressed in a scalar form of $\alpha u + \beta v$"

Solution

w is a linear combination of u and v implies there exist $\alpha, \beta \in \mathbb{R}$ such that

$w = \alpha u + \beta v$

$\Rightarrow (1, K, 4) = \alpha(1, 2, 3) + \beta(2, 3, 1)$

$\Rightarrow (1, K, 4) = (\alpha + 2\beta, 2\alpha + 3\beta, 3\alpha + \beta)$

$\Rightarrow \alpha + 2\beta = 1 \ldots (1)$

$2\alpha + 3\beta = K$

$3\alpha + \beta = 4$

$\alpha = 1 - 2\beta$, $3(1 - 2\beta) + \beta = 4$

$\alpha = 1 + \frac{2}{5}$ $\Rightarrow 3 - 6\beta + \beta = 4$

$-5\beta = 1$

$\alpha = \frac{5+2}{5}$ $\beta = -1/5$

$\alpha = 7/5$

$\Rightarrow 2(7/5) + 3(-\frac{1}{5}) = K$

$\frac{14}{5} - \frac{3}{5} = K \Rightarrow K = \frac{14-3}{5}$

$K = 11/5$

(3) Express $v = t^2 + 4t - 3$ as a linear combination of polynomials $u_1 = t^2 - 2t + 5$, $u_2 = 2t^2 - 3t$, $u_3 = t + 3$

Solution

It implies we to obtain (α, β, γ) such that $v = \alpha u_1 + \beta u_2 + \gamma u_3$

$\Rightarrow t^2+4t-3 \equiv \alpha(t^2-2t+5)+\beta(2t^2-3t),$
$\gamma(t+3)$

$\Rightarrow t^2+4t-3=(\alpha+2\beta)t^2+(-2\alpha-3\beta+\gamma)t + (5\alpha+3\gamma)$

$\Rightarrow \alpha+2\beta=1 \cdots\cdots (1)$

$-2\alpha-3\beta+\gamma=4 \cdots\cdots (2)$

$5\alpha+3\gamma=-3 \cdots\cdots (3)$

$\alpha=-3,\ \beta=2,\ \gamma=4$

$\therefore V=-3u_1+2u_2+4u_3$

Linear Span

Definition: If V over $\mathbb{K}$ and $v_i \in V$ where $(v_i = v_1+v_2 \cdots v_m)$. There is a span if any element $u \in V$ is a linear combination of $(v_1+v_2+\cdots v_m)$ such that u can be expressed as:

$u=\alpha_1 v_1+\alpha_2 v_2 \cdots \alpha_m v_2$

$$u=\sum_{i=1}^{m}\alpha_i v_i$$

Examples

(1) The set of vectors; $B=\{e_1,e_2,e_3\}$ where $e_1=(1,0,0)$, $e_2=(0,1,0)$ and $e_3=(0,0,1)$ B spans $\mathbb{R}^3$

Since for any vector $u\in\mathbb{R}^3$; $u=(a,b,c)$, $a,b,c\in\mathbb{R}$

$(a,b,c)=(a,0,0)+(0,b,0)+(0,0,c)$
$=a(1,0,0)+b(0,1,0)+c(0,0,1)$

$\therefore (a,b,c)=ae_1+be_2+ce_3$

(2) Show that the vectors $u_1=(1,1,1)$, $u_2=(1,2,3)$, $u_3=(1,5,8)$ spans $\mathbb{R}^3$.

Solution

For any vector $u\in\mathbb{R}^3$, can we express u as a linear combination of u_1,u_2,u_3?

$u\in\mathbb{R}^3 \Rightarrow u=(a,b,c)$ where $a,b,c\in\mathbb{R}$.

Let α,β,γ be scalars;

$u=\alpha u_1+\beta u_2+\gamma u_3$

$(a,b,c)=\alpha(1,1,1)+\beta(1,2,3)+\gamma(1,5,8)$

$\Rightarrow (a,b,c)\equiv \alpha+\beta+\gamma+\alpha+2\beta+5\gamma+\alpha+3\beta+8\gamma$

$\alpha+\beta+\gamma=a$
$\alpha+2\beta+5\gamma=b$
$\alpha+3\beta+8\gamma=c$

Linear Subspaces

If V over $\mathbb{K}$, W is a subspace of V if

(i) ~~W is not empty~~ $\vec{0}\in W$

(ii) For any $u,v\in W$; $\alpha u+\beta v\in W$

$W=\{(a,b,c)\in\mathbb{R}^3;\ a=b=c\}$

(i) $\vec{0}=(0,0,0)\in\mathbb{R}^3$

$\vec{0}\in W$ since $0=0=0$

(ii) For any $u,v\in W$;

$W\Rightarrow u=(a_1,b_1,c_1)\in\mathbb{R}^3$ such that $a_1=b_1=c_1$

$v\in W \Rightarrow v=(a_2,b_2,c_2)\in\mathbb{R}^3$ such that $a_2=b_2=c_2$

$\alpha,\beta\in\mathbb{R}$;

$\alpha u+\beta v=\alpha(a_1,b_1,c_1)+\beta(a_2,b_2,c_2)$
$\equiv(\alpha a_1+\beta a_2;\ \alpha b_1+\beta b_2,\ \alpha c_1+\beta c_2)$

$\alpha u+\beta v\in\mathbb{R}^3$;

Since for $a_ib_i\in\mathbb{R}$; $i=1,2,3$;

$\alpha a_1+\beta a_2,\ \alpha b_1+\beta b_2,\ \alpha c_1+\beta c_2\in\mathbb{R}$

For every, u, v, w and constant α, β, and γ $\alpha u + \beta v + \gamma w = 0$ } linearly dependent

$\alpha a_1 + \beta a_2 = \alpha b_1 + \beta b_2 = \alpha c_1 + \beta c_2$

from $a_1 = b_1 = c_1$ and $a_2 = b_2 = c_2$

$\therefore \alpha u + \beta v \in W$

6th May, 2024

(2) Show that W is not a subspace of $V = \mathbb{R}^3$ over $\mathbb{R}$ if $W = \{(a, b, c) ; a^2 + b^2 + c \leq 1\}$

Solution

(1) Conditions:

(i) $\vec{0} \in W$

(ii) if $u, v \in W$; $\alpha, \beta \in \mathbb{R}$, then $\alpha u + \beta v = W$

Now; $\vec{0} = (0,0,0) \in W$

$0^2 + 0^2 + 0^2 \leq 0 < 1$

$\therefore \vec{0} \in W$

But consider vectors $u = (1,0,0)$ and $v = (0,1,0)$

Claim:

$u \in W$ and $v \in W$

$u = (1,0,0)$

$1^2 + 0^2 + 0^2 = 1 \leq 1$

$u \in W$

$v = (0,1,0) = 0^2 + 1^2 + 0^2 = 1 \leq 1$

$\Rightarrow v \in W$

Let $\alpha = 1 = \beta$

$\alpha u + \beta v = u + v$

$= (1,0,0) + (0,1,0) = (1,1,0)$

but $1^2 + 1^2 + 0^2 = 2 \nleq 1$

$\Rightarrow u, v \in W$ but $\alpha u + \beta v \notin W$

Where W is not a vector subspace of W.

(3) If U, W are subspaces of linear vector span V over a field $\mathbb{K}$; then $U \cap W$ is also a vector space V

Linear Dependance and Independence

V over $\mathbb{K}$:

$u_1, u_2 \dots u_m \in V$ is said to be linear dependent, if there exists scalars, $\alpha_1, \alpha_2 \dots \alpha_m \in \mathbb{K}$ such that

$\alpha_1 u_1 + \alpha_2 u_2 + \dots \alpha_m u_m = 0$

not all $\alpha = 0$

if $\alpha_j \neq 0$

$$u_j = -\left(\frac{\alpha_1 u_1}{\alpha_j} + \frac{\alpha_2 u_2}{\alpha_j} + \frac{\alpha_3 u_3}{\alpha_j} + \dots \frac{\alpha_{j-1} u_{j-1}}{\alpha_j} + \dots \frac{\alpha_{j+1} u_{j+1}}{\alpha_j} + \dots \frac{\alpha_m u_m}{\alpha_j}\right)$$

$u_j = \beta_1 u_1 + \beta_2 u_2 + \beta_3 u_3$

For linear independent

For any $\alpha_1 \alpha_2 \dots \alpha_m u_m = 0$

$\alpha_1 = \alpha_2 = \alpha_3 \dots \alpha_m = 0$

Example

$u = (1, -3)$, $v = (-2, 6)$

Solution

$\alpha u + \beta v = \alpha(1, -3) + \beta(-2, 6)$

$= (\alpha - 2\beta, -3\alpha + 6\beta)$

$\alpha u + \beta v = 0$

$\Rightarrow \alpha - 2\beta = 0 \dots (i)$

$-3\alpha + 6\beta = 0 \dots (ii)$

no non zero solution for α, β

$\therefore$ u & v are linearly dependent

since $\alpha u + \beta v = 0$, $\alpha \neq \beta \neq 0$

In particular,

$v = -2u$

$(-2,6) = -2(1,-3) = -2u$

(2) $u = (1,2,-3)$, $v = (4,5,-6)$

for any scalar α, β,

$\Rightarrow \alpha + 4\beta = 0$ ......(i)

$2\alpha + 5\beta = 0$ ......(ii)

$-3\alpha - 6\beta = 0$ ......(iii)

* $u = \begin{bmatrix} 1 & 3 & -4 \\ 5 & 0 & -1 \end{bmatrix}$ $v = \begin{bmatrix} -4 & -12 & 16 \\ -20 & 0 & 4 \end{bmatrix}$

$\begin{bmatrix} 1 & 3 & -4 \\ 5 & 0 & -1 \\ -4 & -12 & 16 \\ -20 & 0 & 4 \end{bmatrix} \xrightarrow[4R_2+R_4]{4R_1+R_3} \begin{bmatrix} 1 & 3 & -4 \\ 5 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\Downarrow$ linearly dependent

$u = 1, -3$, $v = -2, 6$

* $\begin{pmatrix} 1 & -3 \\ -2 & 6 \end{pmatrix} \xrightarrow[2R_1+R_2]{} \begin{pmatrix} 1 & -3 \\ 0 & 0 \end{pmatrix} \Rightarrow$

* (3) $u = (1,2,-3)$, $v = (4,5,-6)$

$\begin{pmatrix} 1 & 2 & -3 \\ 4 & 5 & -6 \end{pmatrix} \xrightarrow[-4R_1+R_2]{} \begin{pmatrix} 1 & 2 & -3 \\ 0 & -3 & 6 \end{pmatrix}$

$\Rightarrow \frac{1}{3}R_2 \begin{pmatrix} 1 & 2 & -3 \\ 0 & 1 & -2 \end{pmatrix} \to$ independent

$f, g \in C^1[a,b]$

f & g are linearly dependent iff

$W(f,g) = 0$

$W(f,g) = \begin{vmatrix} f & g \\ f' & g' \end{vmatrix}$

(5)

Show that f, g, and h are linearly dependent if $f(t) = e^t$, $g(t) = e^{2t}$, $h(t) = t$

Solution

$$W(f,g,h) = \begin{vmatrix} f & g & h \\ f' & g' & h' \\ f'' & g'' & h'' \end{vmatrix}$$

$$= \begin{vmatrix} e^t & e^{2t} & t \\ e^t & 2e^{2t} & 1 \\ e^t & 4e^{2t} & 0 \end{vmatrix}$$

$$= e^t(-4e^{2t}) - e^{2t}e^t + t(4e^{3t} - 2e^{3t})$$

$$= -4e^{3t} + e^{3t} + t(2e^{3t})$$

$$= -3e^{3t} + 2te^{3t}$$

$$e^{3t}(2t-3)$$

For all $t \neq \frac{3}{2}$, $W(f,g,h) \neq 0$

hence f, g, h are linearly independent

(6) Suppose u, v, w are linearly independent vector, show that $S = \{u+v-2w, u-v-w\}$

for any scalar

$$\alpha(u+v-2w) + \beta(u-v-w) = 0$$

$$\Rightarrow (\alpha+\beta)u + (\alpha-\beta)v + (-2\alpha-\beta)w = 0$$

$\gamma = \alpha + \beta$

$\tau = \alpha - \beta$

$\xi = -2\alpha - \beta$

$$\Rightarrow \gamma u + \tau v + \xi w = 0$$

Since u, v, w are linearly indepent it implies that

$$\gamma = \tau = \xi = 0$$

$$\alpha + \beta = 0$$

$$\alpha - \beta = 0$$

$$-2\alpha - \beta = 0$$

$$\Rightarrow \alpha = 0 = \beta$$

Hence $u+v-2w$ & $u-v-w$ are linearly independent

Basis and Dimension

Example

(1) Determine whether or not the following vectors for a basis for $\mathbb{R}^3$

(a) $(1,1,1)$, $(1,2,3)$, $(2,-1,1)$

Solution

They are basis if:

(i) linearly independent

(ii) span or generate $\mathbb{R}^3$

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 2 & -1 & 1 \end{pmatrix} \xrightarrow[-2R_1+R_3]{-R_1+R_2} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & -3 & -1 \end{pmatrix}$$

$$\xrightarrow{3R_2+R_3} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 5 \end{pmatrix} \xrightarrow{\frac{1}{5}R_3} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

(2) Find the dimension and a basis of the Subspace W of $P_3(t)$ spanned by

$u = t^3 + 2t^2 - 3t + 4$, $v = 2t^3 + 5t^2 - 4t + 7$, $w = t^3 + 4t^2 + t + 2$

$$\begin{pmatrix} 4 & -3 & 2 & 1 \\ 7 & -4 & 5 & 2 \\ 2 & 1 & 4 & 1 \end{pmatrix} \xrightarrow{\frac{1}{4}R_1} \begin{pmatrix} 1 & -3/4 & 1/2 & 1/4 \\ 7 & -4 & 5 & 2 \\ 2 & 1 & 4 & 1 \end{pmatrix}$$

$$\xrightarrow{-7R_1+R_2} \begin{pmatrix} 1 & -3/4 & 1/2 & 1/4 \\ 0 & 5/4 & 3/4 & 1/4 \\ 0 & 5/2 & 3 & 1/2 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & \frac{-2}{4} & 1/2 & 1/4 \\ 0 & 5/4 & 3/2 & 1/4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Linear Algebra Slides

MTH 202

Linear Algebra I

Contents

Definitions and some examples of Linear vector spaces and subspaces.

Definitions and examples of linear combination of vectors and span of linear vector space.

Definition and examples of linear dependency or independency of vectors in a linear vector space.

Basis and dimension of linear vector subspaces.

Linear dependency, system of linear equations and matrices.

§1 Linear vector spaces and subspaces

1.1 Definition (Linear vector spaces)

Let V be a nonempty set and let $\mathbb{K}$ be a field of scalars; V is said to be a linear vector space over $\mathbb{K}$ if V is endowed with operations addition, $+$ and scalar multiplication $\cdot$, i.e. $(V, +, \cdot)$ such that the following axioms are satisfied for all $u, v, w \in V$; $\alpha, \beta \in \mathbb{K}$;

A_1 : $u+v \in V$ — closure

A_2 : $u+v = v+u \in V$ — commutativity

A_3 : $(u+v)+w = u+(v+w)$ — associativity

A_4 : There is a vector $\underline{0} \in V$, called zero vector, such that

$$u+0 = 0+u = u .$$

A_5 : For each $u \in V$, there is a vector $-u \in V$ such that

$$u+(-u) = (-u)+u = 0$$

M_1 : $\alpha(u+v) = \alpha u + \alpha v$, for any scalar $\alpha \in \mathbb{K}$

M_2 : $(\alpha+\beta)u = \alpha u + \beta u$, for any scalars $\alpha, \beta \in \mathbb{K}$

M_3 : $(\alpha\beta)u = \alpha(\beta u)$, for any scalars $\alpha, \beta \in \mathbb{K}$

M_4 : $1 \cdot u = u$, for the unit scalar $1 \in \mathbb{K}$.

Examples

1 (a) $\mathbb{R}^n$ over $\mathbb{R}$

(b) $\mathbb{C}^n$ over $\mathbb{C}$.

2. $\mathbb{P}_n(\mathbb{R})$ — Space of all polynomials with degree less than or equal to n with coefficients in $\mathbb{R}$.

3. $M_{m,n}(\mathbb{R})$ — Space of all $m \times n$ — matrices with real entries over ~~R~~ $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$).

4. $\mathbb{F}(\mathbb{R})$ — the space of all functions, $f : \mathbb{R} \to \mathbb{R}$ with pointwise addition and scalar multiplication.

1-2. Linear combinations

Definition: Let V be a linear vector space over a field $\mathbb{K}$, a vector $v \in V$ is a linear combination of vectors $u_1, u_2, u_3, \dots, u_n \in V$ if there exist scalars $\alpha_1, \alpha_2, \alpha_3, \dots \alpha_n \in \mathbb{K}$ such that

$$V = \alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3 + \dots + \alpha_n u_n .$$

Examples

1) Express $v = (1, -2, 5) \in \mathbb{R}^3$ as a linear combination of the vectors $u_1 = (1,1,1)$, $u_2 = (1,2,3)$ and $u_3 = (2,-1,1)$.

$v = -6u_1 + 3u_2 + 2u_3$

2. If $w = (1, k, 4)$, $u = (1, 2, 3)$ and $v = (2, 3, 1) \in \mathbb{R}^3$, find k so that w is a linear combination of u and v.

$\alpha = \frac{k-2}{3}$, $\beta = \frac{2k-5}{3}$

~~Impossible~~.

~~3~~. Express $v = (2, -5, 3) \in \mathbb{R}^3$ as a linear combination of the vectors $u_1 = (1, -3, 2)$, $u_2 = (2, -4, -1)$ and $u_3 = (1, -5, 7)$

$\beta - \gamma = \frac{1}{8}$, $\beta - \gamma = \frac{1}{5}$ Impossible.

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4. Express $V = t^2+4t-3$ as a linear combination of the polynomials $u_1 = t^2-2t+5$, $u_2 = 2t^2-3t$, $u_3 = t+3$

$V = -3u_1+2u_2+4u_3$.

5. Let $D = \begin{bmatrix} 4 & 7 \\ 7 & 9 \end{bmatrix}$, $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

$B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 1 \\ 4 & 5 \end{bmatrix}$

Express D as a linear combination of A, B & C.

$D = 2A+3B-C$.

1-3 Linear Span

Let V be a linear vector space over a field $\mathbb{K}$. Vectors $u_1, u_2, \dots, u_m$ in V are said to span or generate V if every $v \in V$ is a linear combination of the vectors $u_1, u_2, \dots, u_m$.

The collection of all such linear combinations, denoted by $span(u_1, u_2, \dots, u_m)$ is called a linear span of V.

Examples

1) Let $e_1 = (1,0,0)$, $e_2 = (0,1,0)$, $e_3 = (0,0,1)$.
$\{e_1, e_2, e_3\}$ is a span of $\mathbb{R}^3$.

2) Let $w_1 = (1,1,1)$, $w_2 = (1,1,0)$, $w_3 = (1,0,0)$
$\{w_1, w_2, w_3\}$ is a linear span of $\mathbb{R}^3$.

3) Show that the vectors $u_1 = (1,1,1)$, $u_2 = (1,2,3)$, $u_3 = (1,5,8)$ spans $\mathbb{R}^3$.

1.4 Subspaces

Let V be a linear vector space over a field $\mathbb{K}$ and let W be a subset of V. W is said to be a subspace of V if W is itself a linear vector space over $\mathbb{K}$ with respect to the operations of vector addition and scalar multiplication on V.

W is a linear subspace of V if

(i) $0 \in W$ (ii) for every $u, v \in W$; $\alpha, \beta \in \mathbb{K}$; $\alpha u + \beta v \in W$.

Examples

(1) show that W is a subspace of $V = \mathbb{R}^3$ if

(a) $W = \{(a,b,c) : a = b = c\}$ (b) $W = \{(a,b,0) ; a, b \in \mathbb{R}\}$

(c) $W = \{(a,b,c) ; ab = 0\}$

2) show that W is not a subspace of $V = \mathbb{R}^3$ if

(a) $W = \{(a,b,c) : a^2+b^2+c^2 \le 1\}$ (b) $W = \{(a,b,c) : a \ge 0\}$

3) Let V be the linear vector space of functions $f : \mathbb{R} \to \mathbb{R}$. show that W is a subspace of V where

$$W = \{f : f(-x) = -f(x) \quad \forall x \in \mathbb{R}\}$$

4) Let V be the linear vector space of functions

$$f : X \subseteq \mathbb{R} \to X$$

show that W is a subspace of V where

$$W = \{f : f \text{ is continuous on } X\}$$

Ex.

show ~~that~~ ~~Let~~ if V be a linear vector space over a field $\mathbb{K}$, and Let U, W be subspaces of V, then $U \cap W$ is also a subspace of V.

1.5 Linear Dependence and Independence

Definition: Let V be a linear vector space over field $\mathbb{K}$, vectors $v_1, v_2, ---, v_m \in V$ are said to be linearly dependent if there exist scalars $\alpha_1, \alpha_2, ---, \alpha_m$ not all of them 0, such that

$$\alpha_1 v_1 + \alpha_2 v_2 + --- + \alpha_m v_m = 0 .$$

Vectors $u, v \in V$ are said to be linearly dependent if there exist scalars $\alpha, \beta \in \mathbb{K}$; $\alpha \neq \beta \neq 0$ such that

$$\alpha u + \beta v = 0 .$$

It implies that if $\alpha_m \neq 0$;

$$v_m = \left(\frac{-\alpha_1}{\alpha_m}\right) v_1 + \left(\frac{-\alpha_2}{\alpha_m}\right) v_m + --- + \left(\frac{-\alpha_{m-1}}{\alpha_m}\right) v_{m-1}$$

$$\Rightarrow v_m = \gamma_1 v_1 + \gamma_2 v_m + --- + \gamma_{m-1} v_{m-1}$$

$$\text{where } \gamma_i = \frac{-\alpha_i}{\alpha_m}$$

$\Rightarrow$ v_m is a linear combination of $v_1, v_2, ---, v_{m-1}$.

Since v_m was arbitrarily chosen; then ~~A~~ vectors $v_1, v_2, ---, v_m \in V$ are said to be linearly dependent if one of the vectors can be written as a linear combination of other.

For two vectors; ~~⇒~~ $u, v \in V$ are said to be linearly dependent if one is a constant multiple of the other.

Vectors $v_1, v_2, ---, v_m \in V$ are said to be linearly independent if for any scalars $\alpha_1, \alpha_2, ---, \alpha_m$

$$\alpha_1 v_1 + \alpha_2 v_2 + --- + \alpha_m v_m = 0$$

if and only if $\alpha_1 = \alpha_2 = --- = \alpha_m = 0$.

Example

1) Determine whether or not these vectors ~~are~~ u, v are linearly dependent;

(a) $u=(1,2,-3)$, $v=(4,5,-6)$ no

(b) $u=(1,-3)$, $v=(-2,6)$. Yes

2) Determine whether or not u and v are linearly dependent

(a) $u=2t^2-3t+4$; $v=4t^2-3t+2$ $v=24$.

(b) $u=\begin{bmatrix} 1 & 3 & -4 \\ 5 & 0 & -1 \end{bmatrix}$, $v=\begin{bmatrix} -4 & -12 & 16 \\ -20 & 0 & 4 \end{bmatrix}$, $v=-44$

(3) Let $f, g \in C^1[a,b]$; show that f and g are linearly dependent if and only the Wronskian, $W(f,g)$;

$$W(f,g) = \begin{vmatrix} f & g \\ f' & g' \end{vmatrix} = 0 .$$

(4) Show that ~~f and g~~ f, g and h are linearly independent;

(a) $f(t)=e^t$, $g(t)=\sin t$, $h(t)=t$.

(b) $f(t)=e^t$, $g(t)=e^{2t}$, $h(t)=t$. .

(5) Show that $u=(a,b)$ and $v=(c,d) \in \mathbb{R}^2$ are linearly dependent if and only if $ad-bc=0$.

(6) Suppose u, v, w are linearly independent ~~are linearly independ~~ vectors. Prove that S is linearly independent set if

(a) $S=\{u+v-2w,\ u-v-w\}$

(b) $S=\{u+v-2w,\ u+3v-w,\ v+w\}$

(c) $S=\{u+v,\ u-v,\ u-2v+w\}$ . .

1.6 Basis and Dimension

Definition: A set $S=\{u_1, u_2, \dots, u_n\}$ of vectors is a basis of a linear vector space V if
(i) S is a linearly independent; (ii) S spans V.
A linear vector space V is said to be of finite dimension n or n-dimensional; ~~if~~ $\dim V = n$; if V has a basis with n-elements.

Example

1) Determine whether or not the following vectors form a ~~basis~~ basis for $\mathbb{R}^3$.

(a) $(1,1,1), (1,2,3), (2,-1,1)$
(b) $(1,1,2), (1,2,5), (5,3,4)$

2) Find the dimension and a basis of the ~~sub~~ subspace W of $\mathbb{P}_3[t]$ spanned by

$u = t^3 + 2t^2 - 3t + 4$, $v = 2t^3 + 5t^2 - 4t + 7$,
$w = t^3 + 4t^2 + t + 2$.

Ans: $\dim W = 2$ basis: $\{t^3 + 2t^2 - 3t + 4,\ t^2 + 2t - 1\}$

(3) Establish the ~~lin~~ linear dependence or independence of vectors.

$v_1 = (1,0,-1,2)$, $v_2 = (1,3,1,6)$; $v_3 = (1,5,-1,16)$

(4) Find the dimension and a basis of the subspace W of $M = M_{2\times3}$ spanned by

$A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 1 & 2 \end{bmatrix}$ $B = \begin{pmatrix} 2 & 4 & 3 \\ 1 & 5 & 8 \end{pmatrix}$, $C = \begin{bmatrix} 1 & 2 & 3 \\ 5 & 7 & 6 \end{bmatrix}$

$\dim U = 2$; basis:

$\{[1,2,1,3,1,2], [0,0,1,1,3,2]\}$.

N.B.

$\dim U + \dim W = \dim(U+W) + \dim(U \cap W)$.

5) Let U be the subspace of $\mathbb{R}^5$ generated by $\{(1,-1,-1,-2,0), (1,-2,-2,0,-3), (1,-1,-2,-2,1)\}$ and W ~~the subspace generated by~~ the ~~sub~~ subspace ~~gen~~ generated by $\{(1,-2,-3,0,-2), (1,-1,-3,2,-4)$ and $(1,-1,-2,2,-5)\}$

Find a basis and dimension for

i) $U+W$ (ii) $U \cap W$.

$\dim(U+W) = 4$; Basis: $\{(1,-1,-1,-2,0), (0,1,1,-2,3), (0,0,1,0,-1), (0,0,0,1,\frac{-3}{2})\}$.

$\dim U = 3$ $\{(1,-1,-1,-2,0), (0,1,1,-2,3), (0,0,1,0,-1)\}$

$\dim W = 3$ $\{(1,-2,-3,0,-2), (0,1,0,2,-2), (0,0,1,0,-1)\}$

$\dim U \cap W = 2$ basis $\{(1,-2,-3,0,-2), (0,0,1,0,-1)\}$.

Class Notes on Linear Maps

18/05/2024

LINEAR MAPS / Transformation

$T: V_1 \rightarrow V_2$ Vector Spaces

$u \rightarrow T(u)$

$w \rightarrow T(w)$

$T(u+w) = T(u) + T(w)$

$T(\lambda u) = \lambda T(u)$

where $u, w \in V_1$ & $\lambda \in \mathbb{K}$ scalar field

A linear map (transformation) T from vector space V_1 to another vector space V_2 with the underlying scalar field $\mathbb{K}$ is said to be linear if:

(i) $T(u+w) = T(u) + T(w)$ &

(ii) $T(\lambda u) = \lambda T(u)$.

$u, w \in V_1$ & $\lambda \in \mathbb{K}$

$T: V_1 \longrightarrow V_2$

$T(u+w) = T(u) + T(w)$

$T(\lambda u) = \lambda T(u)$

Equivalently, $T: V_1 \rightarrow V_2$ is a linear map if and only if $T(\lambda_1 u + \lambda_2 w) = \lambda_1 T(u) + \lambda_2 (T(w))$

whenever $u, w \in V_1$ and $\lambda_1, \lambda_2 \in \mathbb{K}$.

Matrix multiplication

$T(v) = Av$ (matrix) gives a linear map

$T: V_1 \longrightarrow V_2$

$T(v) \longrightarrow Av$

$T(v+w) = A(v+w)$

$= Av + Aw = T(v) + T(w)$

$T(\lambda v) = A(\lambda v) = \lambda(Av)$

$= \lambda T(v)$

Suppose $v = 0$

$T(v) = T(0)$

$T(0) = T(0+0)$

$= T(0) + T(0)$

$\Rightarrow T(0) = 0$

- For all linear map $T(0)$ must be zero

If $T(w) = w + u_0$ is not linear, but is linear when $u_0 = 0$

- If a map takes a vector and adds a constant, it can't be linear unless the constant is 0.

For linear map, (matrix) $T(v) = Av$

Example

- Dots products are linear.

$T : \mathbb{R}^3 \to \mathbb{R}$

$T(v) = a \cdot v$,

$v = (v_1, v_2, v_3)$, $a = (1, 3, 4)$

$T(v) = (v_1, v_2, v_3) \cdot (1, 3, 4)$

$= v_1 + 3v_2 + 4v_3$

$= \begin{bmatrix} 1 & 3 & 4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$

$= Av$

$T(v) = Av$; $A \to$ matrix rep. T.

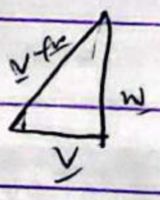
Note

If the output (image) involves squares of products or lengths, v^2, vw, or $\|v\|$, then T is not linear.

Ex: $T(v) = \|v\|$ is not linear

$T(v+w) = \|v+w\| = T(v) + T(w)$

$= \|v\| + \|w\|$



$\|v+w\| \leq \|v\| + \|w\|$

what about $\|-v\|$?, $\lambda = -1$

$\lambda v = -v \Rightarrow \|\lambda v\| = \|-v\|$

$\|-v\| = \|v\|$

$= -\|v\|$ is not linear

Ex: Define the linear map $T : \mathbb{R} \to \mathbb{R}$ by

$T(x) = mx$; where m is a fixed real number.

T linear?

Soln

Let $x, y \in \mathbb{R}$, $\lambda \in K = \mathbb{R}$

$T(x+y) = m(x+y)$

$= mx + my$

$= T(x) + T(y)$

$T(\lambda x) = m(\lambda x) = \lambda(mx)$.

$= \lambda T(x)$

$\therefore T$ is linear.

Q: $T : \mathbb{R} \to \mathbb{R}$, $T(x) = mx + c^{\neq 0}$,

$m, c \in \mathbb{R}$. , T linear? NO

$T(0) = c \neq 0$

Example: $T : \mathbb{R}^2 \to \mathbb{R}^2$

$T(x_1, x_2) = (x_1 + x_2, x_1 - x_2 + 1)$

$T(0, 0) = (0, 1) \neq 0$

14/05/2024

If the linear map is identity, then the matrix representing it should also be identity

Q: Let $T: \mathbb{R}^2 \to \mathbb{R}$ be defined by
$T(x_1, x_2) = x_1^2 + x_2^2$.
Show that T is not a linear even though
$T(0,0) = 0$

Q: Verify that the zero map
$\mathbb{O}: V \to W$ is linear.
$\mathbb{O}(v) = 0$

Q Let $T: P_2(\mathbb{R}) \to P_1(\mathbb{R})$ be defined by $T(a_2x^2 + a_1x + a_0) = 2a_2x + a_1$
Show that T is linear.

Q: Let M_{22} be the set of 2x2 matrices
Show that
$T: M_{22} \to M_{22}$ defined by
$$T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

KERNEL OF A LINEAR MAP

$T: V_1 \to V_2$
$\ker T = \{ v \in V_1 \mid T(v) = 0 \}$

When it's only one 0 vector in the map, a one-to-one mapping is when the $\ker T = 0$

Given a linear map
$T: V_1 \to V_2$,
the kernel of T denoted ker T is given by: $\ker T = \{ v \in V_1 \mid T(v) = 0 \}$

Example: Find ker T where $T: \mathbb{R}^3 \to \mathbb{R}^2$ is defined by
$T(x_1, x_2, x_3) = (x_1 + x_2, x_2 - x_3)$
Soln
$\ker T = \{ x \in \mathbb{R}^3 \mid T(x) = 0 \}$
$= \{ (x_1, x_2, x_3) \in \mathbb{R}^3 \mid (x_1 + x_2, x_2 - x_3) = (0,0) \}$

We must solve

$x_1 + x_2 = 0$

$x_2 - x_3 = 0$

$x_1 + x_2 = 0$

$x_2 - x_3 = 0$

$\in \mathbb{R}$

Let x_3 be k_1, then $x_2 = x_3 = k_1$

$x_1 = -x_2 = -k$.

The solution ~~we're seeking~~ is all triples of $(-k, k, k)$

$\ker T = \{x \in \mathbb{R}^3 \mid x = k(-1,1,1)$

$= \text{span}\{(-1,1,1)\}$

$= \langle\{-1,1,1)\rangle$

Kernel of a linear map is a subspace of the domain.

In the example above, a basis for $\ker T$ is $\{(-1,1,1)\}$.

The dimension of $\ker T$ is called the nullity of T.

This is denoted by $\eta(T)$.

$\eta(T) = 1$

Ex: Calc. $\eta(T)$ for the linear map T:

$T: \mathbb{R}^3 \to \mathbb{R}^2$ defined by

$T(a,b,c) = (a+2b+c, -a+3b+c)$.

Find a basis for $\ker T$.

Solution

$\ker T = \{(a,b,c) \in \mathbb{R}^3 \mid T(a,b,c) = (0,0)\}$

$= \{(a,b,c) \in \mathbb{R}^3 \mid (a+2b+c, -a+3b+c) = (0,0)\}$

$a+2b+c = 0$

$-a+3b+c = 0$

$R_2 = R_2 + R_1$

$a+2b+c = 0$

$0 \quad 5b+2c = 0$

Let $c = k_1$, then $b = -\frac{2}{5}k_1$

$a = -\frac{1}{5}k_1$

The solution is

$(a,b,c) = \frac{k_1}{5}\begin{pmatrix}-1\\-2\\+5\end{pmatrix}$

$(a,b,c) = \frac{k_1}{5}(-1,-2,5)$

$= k(-1,-2,5)$, where $k = \frac{k_1}{5}$

$\ker T = \text{span}\{(-1,-2,5)\}$

$\dim(\ker T) = 1$

i.e. $\eta(T) = 1$

A basis is $\{(-1,-2,5)\}$

② $T: \mathbb{R}^3 \to \mathbb{R}^3$

$T(x_1,x_2,x_3) = (x_1,0,0)$

Determine $\ker T$ and find a basis for $\ker T$.

RANGE OF A LINEAR MAP

Definition: Let $T: V \to W$ be a linear map. The range of T is the set of all possible $y \in W$ such that $y = T(x)$ for some $x \in V$. It is denoted by range T.

\# The range of T is a subspace of W.

Example: Define $T: \mathbb{R}^3 \to \mathbb{R}^3$ by $T(a,b,c) = (a-b+c, 2a+b-c, -a-2b+2c)$.

Det. range T and dim(range T).

Find two vectors in range T and two vectors not in range T.

Soln

Let $y = (y_1, y_2, y_3)$ be in range T. Then, $y = T(a,b,c)$ for some (a,b,c) in the domain $\mathbb{R}^3$

$(y_1, y_2, y_3) = (a-b+c, 2a+b-c, -a-2b+2c)$

$a - b + c = y_1$

$2a + b - c = y_2$

$-a - 2b + 2c = y_3$

$\left.\begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 + R_1 \end{array}\right\}$

$a - b + c = y_1$

$3b - 3c = y_2 - 2y_1$

$-3b + 3c = y_1 + y_3$

$a - b + c = y_1$

$3b - 3c = y_2 - 2y_1$

$0 = -y_1 + y_2 + y_3 \quad R_3 \to R_3 + R_2$

range $T = \{(y_1, y_2, y_3) \in \mathbb{R}^3 \mid y_1 = y_2 + y_3\}$

$(1,0,1) \quad (1,1,0)$

To obtain vectors in range T, fix y_2 & y_3 arbitrarily and then use the condition $y_1 = y_2 + y_3$ to find y_1.

Some vectors in range T are; $(-2,-1,-1)$ and $(0,-1,1)$.

Some vectors not in range T are: $(1,1,1)$ and $(1,0,0)$.

The dim(range T) is 2, since the eqn $-y_1 + y_2 + y_3 = 0$ allows the assignment of arbitrary values to any 2 of the values y_k.

To obtain a basis, we can use $(-2,-1,-1)$ and $(0,-1,1)$ since they are linearly independent in the range and so, dim(range T) = rank T

dim(range T) = 2

infact, any 2 linearly independent vector in range T form a basis for range T.

The dimension of the range of a linear map T is called the rank of T and written R(T)

$R(T) = \dim(\text{Range } T)$

The rank and the nullity of a linear map are related.

rank T + nullity T = dim(domain)

Theorem

If $T: V \to W$ is a linear map and $\dim V = n$, then

$R(T) + \eta(T) = n$ (dimension of the domain of T)

Example 2: Find the nullity of the linear map in example 1.

Sol

$T: \mathbb{R}^3 \to \mathbb{R}^3$

We found that $R(T) = 2$

dimension of the domain $\mathbb{R}^3 = 3 = n$

$\eta(T) = n - R(T) = 3 - 2 = 1$

η

Example 3: Obtain ker T and range T for the linear map $T: \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(x_1, x_2, x_3) = (-x_1 + x_2 + x_3, 2x_1 - x_2, x_1 + x_2 + 3x_3)$

Soln

$T(x) = y$; $T(x_1, x_2, x_3) = (y_1, y_2, y_3)$

$-x_1 + x_2 + x_3 = y_1$

$2x_1 - x_2 = y_2$

$x_1 + x_2 + 3x_3 = y_3$

By successive row reductions

$$\left(\begin{array}{ccc|c} -1 & 1 & 1 & y_1 \\ 0 & 1 & 2 & y_2 + 2y_1 \\ 0 & 0 & 0 & y_3 - 2y_2 - 3y_1 \end{array}\right)$$

To determine ker T, set $y = (0, 0, 0)$

$$\left(\begin{array}{ccc|c} -1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$x_2 + 2x_3 = 0$, Let $x_3 = k$

$x_2 = -2k$ and

$-x_1 + (-2k) + k = 0 \quad \therefore -x_1 - k = 0$

$x_1 = -k$

The solution is $(-k, -2k, k)$

$= k(-1, -2, 1)$

$\ker T = \text{span}\{(-1, -2, 1)\}$

Range $T = \{y \in \mathbb{R}^3 \mid y_3 - 2y_2 - 3y_1 = 0\}$

Fix y_1 and y_2 and express y_3 in terms of y_1 & y_2.

range $T = \{y \in \mathbb{R}^3 \mid y = (s, t, 3s+2t)\}$

$= \{y \in \mathbb{R}^3 \mid y = (s+0, 0+t, 3s+2t\}$

$= \{y \in \mathbb{R}^3 \mid y\, s(1,0,3) + t(0,1,2)\}$

$= \text{Span}\{(1,0,3), (0,1,2)\}$

$\dim(\text{range } T) = \text{rank } T = 2$

dim of domain $T = n = 3$

$\therefore \eta(T) = 3 - 2 = 1$

Example 4: Show that $T: M_{2,1} \to M_{2,1}$ defined by

$$T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

transforms the unit circle $x_1^2 + x_2^2 = 1$ to an ellipse.

Soln

$$T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

The image of $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ under T is

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 2x_1 \\ 3x_2 \end{pmatrix}$$

$y_1^2 = 4x_1^2, \quad y_2^2 = 9x_2^2$

$\frac{y_1^2}{4} = x_1^2, \quad \frac{y_2^2}{9} = x_2^2$

$\frac{y_1^2}{4} + \frac{y_2^2}{9} = x_1^2 + x_2^2 = 1$

$\therefore \frac{y_1^2}{4} + \frac{y_2^2}{9} = 1$

$\therefore$ The image of the unit circle is an ellipse.

Q $T: \mathbb{R}^2 \to \mathbb{R}^2$; $T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

Show that T transforms the circle $x_1^2 + x_2^2 = 1$ to the ellipse $\frac{y_1^2}{a^2} + \frac{y_2^2}{b^2} = 1$.

16th May, 2024

Matrix Representation For Linear Maps

$A_{m\times n} X_{n\times 1} = B_{m\times 1}$

$A(cX+dY) = cAX + dAY$

$\cancel{\equiv}$ $\equiv$

$T(cX+dY) = cT(X) + dT(Y)$

$A_{m\times n}$

$\cancel{\text{dim}}\; T: \mathbb{R}^n \longrightarrow \mathbb{R}^m$

dim(solution space of $AX=0$) + rank $A = n$ = number of columns of A

Example

Find the dimension of the solution space of $AX=0$, where

$$A = \begin{pmatrix} 1 & 2 & -1 & 2 \\ 3 & 1 & 2 & 2 \\ 2 & -1 & 3 & 0 \\ 1 & -2 & 4 & -2 \end{pmatrix}$$

$n = 4$ = number of columns

dim (solution space of $AX=0$)

$= 4 -$ rank A

row reduce A to echelon form:

$$\begin{pmatrix} 1 & 2 & -1 & 2 \\ 0 & -5 & 5 & -4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \frac{2 \text{ zero rows}}{\text{dim (solution space)}} = 2$$

2 non zero rows = rank A

dim (solution space of $AX=0$)

$= 4 - 2 = 2$

Given $T: V \longrightarrow W$

dim $(V) = n$, dim$(W) = m$

Then the matrix representing T is an $m \times n$ matrix

Example

Consider the identity transformation

$T: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$

defined by $T(X) = X$

Let $X = (x)$ where $\mathcal{E}$ is the standard

Ordered basis

$\{(1,0,0), (0,1,0), (0,0,1)\}$

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$x_1\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + x_2\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + x_3\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$T(x) = x$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$I_{3\times 3} X = X$

Example: Projection $P: \mathbb{R}^3 \to \mathbb{R}^3$

$P(x_1, x_2, x_3) = (x_1, x_2, 0)$

Taking $\mathcal{E}$ as in the last example

$$MX = (?)\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix}$$

$$M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Note $P(P(x)) = P(x)$; $P^2(x) = P(x)$

$MM = M$; $M^2 = M$

Example: Consider the differentiation

$D: P_1 \longrightarrow P_0$

defined by

$D(a+bx) = b = b + 0x$

If we use the standard basis

$\mathcal{E} = \{1, x\}$

$$(a+bx)_{\mathcal{E}} = \begin{pmatrix} a \\ b \end{pmatrix}$$

$D: \begin{pmatrix} a \\ b \end{pmatrix} \to \begin{pmatrix} b \\ 0 \end{pmatrix}$

Note $(a+bx)' = b = b+0x$

$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} b \\ 0 \end{pmatrix}$

$M = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $\quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

Observe:

$D(D(a+bx)) = D(b+0x) = 0+0x$

$MM = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

D is represented by a matrix which is nilpotent of exponent

Matrix Representation

$T: V \to W$ (Linear map)

$\dim V = n$

Basis for V: $\{v_1, v_2, \dots v_n\}$

Then, the range of T is compatibility describable in terms of images $T(v_1), T(v_2), \dots, T(v_n)$

Let $x \in V = c_1 v_1 + c_2 v_2 + \dots c_n v_n$

$T(x) = T(c_1 v_1 + c_2 v_2 + \dots + c_n v_n)$

$= c_1 T(v_1) + c_2 T(v_2) + \dots c_n T(v_n)$

Thus $T(x) \in \text{span}\{T(v_1) \dots T(v_n)\}$

Suppose $\dim V = n$, $B = \{v_1, \dots, v_n\}$ is an ordered basis for V,

$\dim W = m$, $B = \{w_1, \dots w_m\}$ is an orderad bas of W

(1) Calculate $T(v_1), T(v_2), \dots T(v_n)$

(2) Find the coordinate vectors

$T(v_1))_{B'}, T(v_2))_{B'} \dots, (T(v_n))_{B'}$

(3) Write the matrix with columns as the column vectors calculated in step 2!

Example: Let $T: (\mathbb{R}^2, B) \to (\mathbb{R}^2, B')$ be defined by $T(x_1, x_2) = (x_1 + 2x_2, x_1 - x_2)$

Find the matrix M representing T when

(a) $B = B'$ = The standard basis

(b) $B = B' = \{(1,2), (3,-1)\}$

(c) $B = \{(1,-1)\ (1,1)\}$

$B' = \{(1,0), (0,-1)\}$

(d) $B = \{(1,-1), (1,1)\}$, $B' = \{(0,-1), (1,0)\}$

Solution

(a) $T(\underset{e_1}{1,0}) = (1+2(0), 1-0) = (1,1)$

$T(0,1) = (0+2(1), 0-1) = (2,-1)$

e_2

$(1,1) = 1(1,0) + 1(0,1)$

$T(e_1) = (1,1)$

$(T(e_1))_{B'} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$(T(e_2))_{B'} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$ $\quad (2,-1) = 2(1,0) + -1(0,1)$

The matrix is

$M = \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}$

Using Any vector from the domain $(3,2)$

$(3,2) = (3+2(2), 3-2) = (7,1)$ by definition of T

$(3,2) = 3(1,0) + 2(0,1)$

$(3,2)_B = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

$T(3,2)_{B'} = M \begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

$\begin{pmatrix} 7 \\ 1 \end{pmatrix} \to 7(1,0) + 1(0,1) = (7,1)$

(b) $T(1,2) = (5,-1)$

$= \frac{2}{7}(1,2) + \frac{11}{7}(3,-1)$

$(T(1,2))_{B'} = \begin{pmatrix} \frac{2}{7} \\ \frac{11}{7} \end{pmatrix}$

$T(3,-1) = (1,4) = \frac{13}{7}(1,2) + \left(\frac{-2}{7}\right)(3,-1)$

$(T(3,-1))_{B'} = \begin{pmatrix} 13/7 \\ -2/7 \end{pmatrix}$

$M = \begin{pmatrix} \frac{2}{7} & \frac{13}{7} \\ \frac{11}{7} & \frac{-2}{7} \end{pmatrix}$

(d) $T(1,-1) = (-1, 2) = -2(0,-1) + -1(1,0)$

$(T(1,-1))_{B'} = \begin{pmatrix} -2 \\ -1 \end{pmatrix}$

$T(1,1) = (3,0) = 0(0,-1) + 3(1,0)$

$(T(1,1))_{B'} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$

$M = \begin{pmatrix} -2 & 0 \\ -1 & 3 \end{pmatrix}$

20th May, 2024

Transition Matrix

We've seen that the matrix associated to a given linear map $T: V \to W$ is dependent on the bases chosen for V and W.

We will try to describe how the matrix associated with T changes when we change the basis in either, or both of the domain in the codomain

Changing The Domain Basis

Illustration, Let $T: \mathbb{R}^3 \to \mathbb{R}^2$ be a linear map given by $T(x,y,z) = (x+2y,\ y-z)$ and let A be the matrix associated with T with respect to the standard bases $\{(1,0,0), (0,1,0), (0,0,1)\}$ and $\{(1,0), (0,1)\}$

$\therefore$

$T(1,0,0) = (1,0) = 1(1,0) + 0(0,1)$

$T(0,1,0) = (2,1) = 2(1,0) + 1(0,1)$

$T(0,0,1) = (0,-1) = 0(1,0) + -1(0,1)$

${}_cA_d = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \end{pmatrix}$

Suppose that B represents T with respect to the bases $\{(1,0,0), (1,1,0), (1,1,1)\}$ and $\{(1,0), (0,1)\}$

$T(1,0,0) = (1,0) = 1(1,0) + 0(0,1)$

$T(1,1,0) = (3,1) = 3(1,0) + 1(0,1)$

$T(1,1,1) = (3,0) = 3(1,0) + 0(0,1)$

${}_cB_{d'} = \begin{pmatrix} 1 & 3 & 3 \\ 0 & 1 & 0 \end{pmatrix}$

Relationship Between A and B

Consider $J: \mathbb{R}^3 \xrightarrow[\text{identity}]{} \mathbb{R}^3$ and let $P = {}_dP_{d'}$ be the matrix representing J with respect to the bases d' in the domain and basis d in the codomain

$J(1,0,0) = (1,0,0) = 1(1,0,0) + 0(0,1,0) + 0(0,0,1)$

$J(1,1,0) = (1,1,0) = 1(1,0,0) + 1(0,1,0) + 0(0,0,1)$

$J(1,1,1) = (1,1,0) = 1(1,0,0) + 1(0,1,0) + 1(0,0,1)$

J is an identity map

${}_dP_{d'} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

Note

$$AP = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$(2\times 3) \times (3 \times 3)$

$$= \begin{pmatrix} 1 & 3 & 3 \\ 0 & 1 & 0 \end{pmatrix} = B$$

$$\begin{tikzcd} V_d \arrow[r, "J"', "\text{identity}"] \arrow[rr, bend left, "T"] & V_d \arrow[r, "T"] & W_c \end{tikzcd}$$

$x \longrightarrow x \longrightarrow T(x)$

$T \circ J = T$

$(T \circ J)(x) = T(x)$

$T(J(x)) = T(x)$

$T(Px) = T(x)$

$APx = T(x) = Bx$

$AP = B$

The matrix P is called the transition matrix

Transition Matrix

Changing the Codomain Basis

Illustration $T: \mathbb{R}^3 \to \mathbb{R}^2$

$T(x,y,z) = (x+2y, -y-z)$

$d: \{(1,0,0), (0,1,0), (0,0,1)\}$

$c: \{(1,0), (1,1)\}$

$T(1,0,0) = (1,0) = 1(1,0) + 0(1,1)$

$T(0,1,0) = (2,1) = 1(1,0) + 1(1,1)$

$T(0,0,1) = (0,-1) = 1(1,0) + (-1)(1,1)$

$$C = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \end{pmatrix}$$

$\mathbb{R}^3 \xrightarrow{I} \mathbb{R}^2 \xrightarrow{I} \mathbb{R}^2$

$I(1,0) = (1,0)$
$= 1(1,0) + 0(0,1)$

$I(1,1) = (1,1)$
$= 1(1,0) + 1(0,1)$

$$Q = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$QC = A$

$Q^{-1} \cdot (QC) = Q^{-1}A$

$(Q^{-1}Q)C = Q^{-1}A$

identity

$C = Q^{-1}A$

Changing both the Domain and the Codomain Bases

$D = Q^{-1}AP$

new matrix

transition matrices

Some Special Matrices

Square matrices: Matrices with number of rows equal the number of columns ($n \times n$) matrices

eg

$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ 2×2 square matrix

$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$ 3×3 square matrix

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

$n \times n$ square matrix

leading diagonal

a_{ij}, $i = j$

(2) Diagonal Matrix

$$\begin{pmatrix} a_{11} & & & 0 \\ & a_{22} & & \\ & & a_{33} & \\ 0 & & & \ddots a_{xx} \end{pmatrix}, \quad a_{ii} \neq 0 \text{ for all } i$$

All elements are zero except those along the leading diagonal.

$a_{ij} = 0$ whenever $i \neq j$

21st May, 2024

* Multiplication of diagonal matrices is commutative:

If A and B are diagonal, then $AB = BA$

(3) Unit Matrix: It is a diagonal matrix with all its diagonal element equals to unity i.e 1. It is usually denoted by I.

Generally, $IA = AI = A$

$I = I^2 = I^3 \dots = I^k \dots$

k is a positive integer

(4) Symmetric Matrices: A symmetric matrix A is a square matrix with elements a_{ij} such that $a_{ij} = a_{ji}$

The elements above and below the leading diagonal are mirror images of each other.

e.g $\begin{pmatrix} 1 & x & y \\ x & 3 & z \\ y & z & 4 \end{pmatrix}$

a_{23}

a_{32}

Every symmetric matrix is its own transpose ~~A = A^T~~ $A = A^T$

(5) Skew - Symmetric Matrices

A skew-symmetric matrix is also a square matrix with element $a_{ij} = -a_{ji}$.

All the elements along the leading diagonal must be zero.

4 $i = j$, $a_{ii} = -a_{ii}$

$a_{ii} + a_{ii} = 0$

$\Rightarrow 2a_{ii} = 0 \Rightarrow a_{ii} = 0$

e.g $\begin{pmatrix} 0 & 2 & 1 \\ -2 & 0 & 3 \\ -1 & -3 & 0 \end{pmatrix}$

Such matrices are the negatives of their own transpose i.e. $A = A^T$

* Any square matrix can be resolved into the sum of a symmetric matrix

$a_{ij} = \frac{1}{2}(a_{ij} + a_{ji}) + \frac{1}{2}(a_{ij} - a_{ji})$

$= \underline{A} + \underset{\sim}{A}$

$\Rightarrow \underline{A}$: symmetric, $\underset{\sim}{A}$ = skew-symmetric

e.g

$A = \begin{pmatrix} 3 & 1 & 2 \\ 2 & 1 & 3 \\ 1 & 2 & 1 \end{pmatrix}$

$a_{ij} = \begin{pmatrix} 3 & 1 & 2 \\ 2 & 1 & 3 \\ 1 & 2 & 1 \end{pmatrix}$, $a_{ji} = \begin{pmatrix} 3 & 2 & 1 \\ 1 & 1 & 2 \\ 2 & 3 & 1 \end{pmatrix}$

$\underline{A} = \frac{1}{2}\begin{pmatrix} 3+3 & 1+2 & 2+1 \\ 2+1 & 1+1 & 3+2 \\ 1+2 & 2+3 & 1+1 \end{pmatrix}$

$\underline{A} = \begin{pmatrix} 3 & \frac{3}{2} & \frac{3}{2} \\ \frac{3}{2} & 1 & \frac{5}{2} \\ \frac{3}{2} & \frac{5}{2} & 1 \end{pmatrix}$

Symmetric matrix

$\underset{\sim}{A} = \frac{1}{2}\begin{pmatrix} 3-3 & 1-2 & 2-1 \\ 2-1 & 1-1 & 3-2 \\ 1-2 & 2-3 & 1-1 \end{pmatrix}$

$$A = \frac{1}{2}\begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & 1 \\ -1 & -1 & 0 \end{pmatrix}$$

$$A = \begin{pmatrix} 0 & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & 0 \end{pmatrix}$$

Skew symmetric

(6) Hermitian Matrices

If A is any matrix of order $m \times n$ whose entries a_{ij} are complex numbers then the complex congugate A^* of A is formed by taking the complex consugate of all the matroces:

If $A = \begin{pmatrix} 1+i & 2 & -i \\ 3 & 1-i & 2+i \end{pmatrix}$

then

$$A^* = \begin{pmatrix} 1-i & 2 & i \\ 3 & 1+i & 2-i \end{pmatrix}$$

Note

$(A^*)^* = A$

A hermitian matrix is a square matrix which is unchanged by ~~the~~ taking the transpose of its complex congugate

i.e A is hermitian if

$$(A^*)^T = A$$

eg If $A = \begin{pmatrix} 1 & 1+i \\ 1-i & 3 \end{pmatrix}$

Then

$$A^* = \begin{pmatrix} 1 & 1-i \\ 1+i & 3 \end{pmatrix}$$

$$(A^*)^T = \begin{pmatrix} 1 & 1+i \\ 1-i & 3 \end{pmatrix} = A$$

* The diagonal elements of a hermitian matrix are always real numbers
* If a square matrix A is such that,

$$(A^*)^T = -A$$

It is called Skew-Hermitian

(7) Orthogonal Matrices

A square matrix A (with real entries) is said to be orthogonal

if $A^T = A^{-1}$

That is

$$A^TA = AA^T = I$$

where I is the unit matrix of the same order as A.

* If A and B are nth order orthogonal matrices then their product is also orthogonal

Suppose A and B are orthogonal then;

$$A^TA = AA^T = I$$

$$B^TB = BB^T = I$$

then

$$(AB)(AB)^T = AB \underbrace{(AB)^T}_{B^TA^T}$$ (Shoe-socks method)

$$= A\underbrace{BB^T}A^T$$

$$= AIA^T = I$$

$$(AB)^TAB = B^T\underbrace{A^TA}B$$

$$= B^TIB = I$$

(8) Triangular Matrices

$$\begin{pmatrix} \text{Not all zero} & 0 \end{pmatrix}$$ lower triangular matrix

27th May, 2024

Matrices & Determinants

Elements in matrix are vectors. (i,j,k) The vector show the position, the number shows the Length. They are represented in rows and columns.

Definition: A set of numbers (real and complex) arranged in a rectangular formation (array or table) having m-rows and n-column(s), enclosed by a square bracket [ ] is called matrix.

$$\circ \longrightarrow \text{row } (i)m$$
$$\downarrow \text{column } (j)n$$

$$A = \{a_{ij}\} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Types of Matrices

* Row & Column matrix
* Null or zero Matrix: All entries are zero

$A = a_{ij} = [0]$ = Null matrix

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

* Square matrix: Number of rows equals to number of columns
* Main or Principal Diagonal Matrix: The Principal diagonal matrix of a square matrix is the ordered set of elements a_{ij}, where $i = j$, extending from upper left-hand corner to the lower right hand corner of the matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{31} \\ a_{21} & a_{22} & a_{32} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

* Diagonal Matrix: It is a square matrix where other elements are zero, except the Leading diagonal matrix

$$A = \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix}$$

* Scalar Matrix: It must be square matrix, must diagonal and the entries of the diagonal matrix are equal.

$$A = \{a_{ij}\} = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$$

* Identity Matrix: They are scalar matrix with leading diagonal being 1.

* Equal Matrix: The rows and colums are the same order, they are not necessarily square matrix.

$$A = \{a_{ij}\}_{mn} \quad \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix}$$
$$B = \{b_{ij}\}_{mn}$$

co1

· Determinants of Matrix

* The determinant of a matrix in a scalar (number) obtained from the elements of matrix by specified operations, which is characteristics of the matrix.
* Determinants are defined only for square matrices
* It is denoted by det A or |A| for a square matrix A.

$a_{11}\begin{bmatrix} a_{22} & \ominus & a_{23} \\ a_{32} & & \end{bmatrix}$

~~Properties of Determinants~~

Minor & cofactors

$$A = \begin{array}{ccc} + & - & + \\ \end{array}$$
$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$a_{11}\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} \rightarrow \text{minor of } a_{11}$$

→ co factor

$$\det A \,//\, |A| = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Properties of Determinant

For any square matrix, |A| satisfies the following

① Interchanging the corresponding rows and columns of a determinant does not change its value, i.e $|A| = |B|$

$$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$$

$$|A| = a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

$$\Rightarrow a_1(b_2c_3 - c_2b_3) - b_1(a_2c_3 - c_2a_3) + c_1(a_2b_3 - b_2a_3)$$

$$B = \begin{bmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{bmatrix}$$

$$|B| = a_2\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} - b_2\begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix} + c_2\begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix}$$

$$|B| = a_2(b_1c_3 - c_1b_3) - b_2(a_1c_3 - c_1a_3) + c_2(a_1b_3 - b_1a_3)$$

$$|A| = a_1b_2c_3 - a_1c_2b_3 - b_1a_2c_3 + b_1c_2a_3 + c_1a_2b_3 - c_1b_2a_3$$

$$|B| = a_2b_1c_3 - b_2a_1c_3 + c_1a_3b_2 + c_2a_1b_3 - c_2b_1a_3$$

② If two rows and two columns of a determinant are interchanged, the sign of the determinant is changed but the absolute value remains unchanged.

$$A = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}; \quad B = \begin{vmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$-|B| = |A|, \text{ or } |A| = -|B|$$

③ If every element of a row or column of a determinant is zero, the value of the determinant is zero.

$$A = \begin{vmatrix} 0 & 0 & 0 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} . \begin{vmatrix} 0 & b_1 & c_1 \\ 0 & b_2 & c_3 \\ 0 & b_3 & c_3 \end{vmatrix}$$

$$|A| = 0$$

(4) If two rows or two columns of a determinant are identical, the value of the determinant is zero

$$|A| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$|A| = a_1(b_1c_3 - c_1b_3) - b_1(a_1c_3 - c_1a_3) + c_1(a_1b_3 - b_1a_3)$$

(5) If every element of a row or column of a determinant is multiplied by the same constant 'K', the value of the determinant is multiplied by that constant.

$$|A| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, \quad |B| = \begin{vmatrix} ka_1 & kb_1 & kc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$|B| = K|A|$$

(6) The value of a determinant is not changed if each element of any row or column is added to (or ~~subtra~~ substracted from) a constant multiple of the corresponding element of another row or column

Considering a matrix given as

$$|A| = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}, \quad |B| = \begin{vmatrix} a_1+ka_2 & b_1+kb_2 & c_1+kc_2 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$|B| = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} + K\begin{bmatrix} a_2 & b_2 & c_2 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$$

$$= |A| + K(0) = |A|$$

(7) The determinant of a diagonal matrix is equal to the product of its diagonal elements

$$A = \begin{bmatrix} a_1 & 0 & 0 \\ 0 & b_2 & 0 \\ 0 & 0 & c_3 \end{bmatrix} \Rightarrow |A| = a_1b_2c_3$$

(8) The determinant of the product of two matrices is equal to the product of the determinants of the two matrices, that's

$$AB = |A|\,|B|$$

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \quad B = \begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix}$$

$$|AB|$$

28th May, 2024

(9) The determinant in which each element in any row, or column consist of two terms, then the determinant can be expressed on the sum of their other determinant

$$\begin{vmatrix} a_1+\alpha_1 & b_1 & c_1 \\ a_2+\alpha_2 & b_2 & c_2 \\ a_3+\alpha_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} \alpha_1 & b_1 & c_1 \\ \alpha_2 & b_2 & c_2 \\ \alpha_3 & b_3 & c_3 \end{vmatrix}$$

$$\begin{vmatrix} a_1+\alpha_1 & b_1+\beta_1 & c_1 \\ a_2+\alpha_2 & b_2+\beta_2 & c_2 \\ a_3+\alpha_3 & b_3+\beta_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} \alpha_1 & b_1 & c_1 \\ \alpha_2 & b_2 & c_2 \\ \alpha_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} a_1 & \beta_1 & c_1 \\ a_2 & \beta_2 & c_2 \\ a_3 & \beta_3 & c_3 \end{vmatrix} + \begin{vmatrix} \alpha_1 & \beta_1 & c_1 \\ \alpha_2 & \beta_2 & c_2 \\ \alpha_3 & \beta_3 & c_3 \end{vmatrix}$$

* How many sum will $\begin{vmatrix} a_1+\alpha_1 & b_1+\beta_1 & c_1+\gamma_1 \\ a_2+\alpha_2 & b_2+\beta_2 & c_2+\gamma_2 \\ a_3+\alpha_3 & b_3+\beta_3 & c_3+\gamma_3 \end{vmatrix}$ have?

* Let $f(t) = \begin{vmatrix} \cos t & t & 1 \\ 2\sin t & t & 2t \\ \sin t & t & t \end{vmatrix}$ then

$$\lim_{t \to 0} \frac{f(t)}{t^2} = ?$$

* If $f(x) = \begin{vmatrix} 0 & x-a & x-b \\ x+a & 0 & x-c \\ x+b & x+c & 0 \end{vmatrix}$, then

$f(0) = ?$ a/

* Find the value of θ that satisfies

$$\begin{bmatrix} 1 & 1 & \sin 3\theta \\ -4 & 3 & \cos 2\theta \\ 7 & -7 & -2 \end{bmatrix} = 0$$

* Minor of the element a_{ij} of the determinant of matrix A is the determinant obtained by deleting ith and jth column, and it is denoted by M_{ij}

a_{ij}

* Cofactor of an element is given by

$A_{ij} = (-1)^{i+j} \cdot M_{ij}$ ⇒ cofactor

Symmetric Matrix:

A square matrix is called symmetric matrix if $A = A^T$

A^T = Changing the row for column or vice versa.

Examples

$$A = \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix} = A^T = \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix}$$

Skew Symmetric Matrix: A square matrix is skew symmetric matrix if $A = -A^T$

E.g

$$A = \begin{bmatrix} 0 & -4 & 1 \\ 4 & 0 & -3 \\ -1 & 3 & 0 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 0 & 4 & -1 \\ -4 & 0 & -3 \\ 1 & 3 & 0 \end{bmatrix}$$

diagonal → skew diagonal

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

Singular Matrix: $\det A = 0$

Adjoint of a Matrix: Let $A = [a_{ij}]$ be a square matrix of order $n \times n$ and $C(c_{ij})$ is a matrix obtained by replacing each element a_{ij} by its corresponding (cofactor)^T

cofactor (a_{ij}), then $(C_{ij})^{t}$ is called the adjoint of A. It is written as adj. A.

$\text{adj}.A = \text{Cofactor}(A)^{T}$

E.g

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{minor} = d$$

$$\text{Cofactor } A = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$$

$$A^{T} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

* Find the adj. A

$$A = \begin{bmatrix} 2 & 1 & 1 \\ -5 & -3 & 0 \\ -1 & 1 & -1 \end{bmatrix}$$

$$\text{Cofactor} = \begin{bmatrix} 3 & -5 & -2 \\ & & \end{bmatrix}$$

Inverse of a Matrix

$$A^{-1} = \frac{1}{\det A} \cdot \text{adj } A$$

* Find the inverse of

$$A = \begin{pmatrix} 1 & -3 \\ -2 & 6 \end{pmatrix} \Rightarrow \text{does not exist}$$

* Find the inverse of

$$A = \begin{pmatrix} 3 & -2 \\ 4 & 7 \end{pmatrix}$$

$$A^{-1} = \begin{pmatrix} 7 & -4 \\ 2 & 3 \end{pmatrix} \div 29 = \begin{pmatrix} \frac{7}{29} & -\frac{4}{29} \\ \frac{2}{29} & \frac{3}{29} \end{pmatrix}$$

* Find a

$$\begin{vmatrix} a^2+2a & 2a+1 & 1 \\ 2a+1 & a+1 & 1 \\ 3 & 3 & 1 \end{vmatrix} = (a-1)^3$$

* Show that

$$A = \begin{vmatrix} \operatorname{cosec}^2\theta & \cot^2\theta & 1 \\ \cot\theta & \operatorname{cosec}^2\theta & -1 \\ 42 & 40 & 2 \end{vmatrix} = 0$$

30th May, 2024

~~System of Linear Equation~~

* find the determinant $\begin{vmatrix} 2i & 3 \\ 4 & -i \end{vmatrix} = -10.$

* Let $A = \begin{vmatrix} Ax & x^2 & 1 \\ By & y^2 & 1 \\ Cz & z^2 & 1 \end{vmatrix}$ and

$$A_1 = \begin{vmatrix} A & B & C \\ x & y & z \\ zy & zx & xy \end{vmatrix}, \text{ then}$$

(a) $A_1 = -A$
(b) $A \neq A_1$
(c) $A - A_1 = 0$ ✓
(d) none of the above

* Find x and y if $\begin{vmatrix} 4i & i^3 & 2i \\ 1 & 3i^2 & 4 \\ 5 & -3 & i \end{vmatrix} = x + iy$

Ans: $x = 11 \quad y = 52$

* Find K such that the given matrix is singular

$$\begin{vmatrix} 1 & 1 & -2 \\ 3 & -1 & 1 \\ K & 3 & -6 \end{vmatrix} = 0$$

* Matrix $[a_{ij}]$ is a row matrix if
(a) $i = 1$ (b) $j = 1$ (c) $m = 1$ ✓ (d) $n = 1$

* Matrix $A = [a_{ij}]$, is an identity matrix if
(a) $\forall i = j, a_{ij} = 0$ (b) $\forall i = j, a_{ij} = 1$
(c) $\forall i \neq j, a_{ij} = 0$ (d) B and C ✓

* Matrix $[c_{ij}]$ is rectangular if
(a) $i \neq j$ (b) $i = j$ (c) $m = n$ (d) $m \neq n$ ✓

* $A = [a_{ij}]$ is scalar matrix if
(a) $\forall i = j, a_{ij} = 0$ (b) $\forall i = j, a_{ij} = k$
(c) $\forall i \neq j, a_{ij} = k$ (d) $\forall i \neq j, a_{ij} = 0$

System of Linear Equation

* Solution of linear equation by matrices

$$\left.\begin{aligned} &a_{11}x_1 + a_{12}x_2 + \dots a_{1n}x_n \\ &a_{21}x_1 + a_{22}x_2 + \dots a_{2n}x_n \\ &\vdots \\ &a_{n1}x_1 + a_{n2}x_2 + \dots a_{nn}x_n \end{aligned}\right\} (1)$$

Putting eqn (1) in matrix form;

$$\begin{bmatrix} a_{11} & a_{12} \dots a_{1n} \\ a_{21} & a_{22} \dots a_{2n} \\ a_{n1} & a_{n2} \dots a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$\Rightarrow AX = B$

$X = A^{-1}B$

where $A^{-1} = \frac{1}{\det A} \cdot \text{adj} A$

$\therefore X = \frac{1}{\det A} \cdot \text{adj} A \cdot B$

Crammer's rule

$x_1 = \frac{|Ax_1|}{|A|}, \; x_2 = \frac{|Ax_2|}{|A|}, \; x_n = \frac{|Ax_n|}{|A|}$

* If $[a_{ij}]$ and $[b_{ij}]$ are of the same order and $a_{ij} = b_{ij}$, then the matrix will be
(a) Singular (b) Null (c) Unequal (d) equal ✓

* If A and B are symmetric, then $AB =$
(a) BA (b) $A^T B^T$ ✓ (c) $B^T A^T$ ✓ (d) None of the above

* If $A = [a_{ij}]$, $B = [b_{ij}]$ of order $m \times n$, then $A + B = [a_{ij} + b_{ij}]$ is of order
(a) $m \times n$ ✓ (b) $m \times m$ (c) $n \times n$ (d) None of the above

* Using crammer's rule, find the unknowns
$x - 2y - 2z = 3$
$2x - 4y + 4z = 1$
$3x - 3y - 3z = 4$

$\Delta = -24$
$\Delta x = 8$
$\Delta y = 25$
$\Delta z =$

3rd June, 2024

* Find the solution of the following system of matrices

$x+y-2z = 3$

$3x+y+z = 0$

$3x+3y-6z = 0$

Ans: The solution does not exist

(a) $\{(1,1,1)\}$ (b) $\{(1,0,1)\}$

(c) $\{(7,-3,1)\}$ (d) $\{(1,2,-3)\}$

Rank and Nullity of Matrices

Rank: Rank of a matrix is the number of linearly independent rows or columns in the matrix.

Method to Determine Rank of Matrix

* Transform the given matrix into its Row-Echelon form (REF). In REF, the maximum number of linearly independent row/columns is equal to the number of non-zero rows in its echelon form.

* Non-zero row is basically a row that contains at least one non-zero value

* Perform element row operation such that the element in the leading diagonal goes to 1 and the element(s) below the leading diagonal goes to zero.

Example leading diagonal

* $A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 4 & 8 \end{bmatrix} \rightarrow R_2 = 2R_1 - R_2$

REF $\Rightarrow \begin{bmatrix} 1 & 2 & 4 \\ 0 & 0 & 0 \end{bmatrix}$ REF

$r(A) = 1$ (rank)

Nullity (no. of column)

$n(A) = n(C) - r(A)$

$n(A) = 3-1$

$n(A) = 2$

* Find the rank and nullity of the given matrix

$\begin{bmatrix} 1 & 1 & 0 & -2 \\ 2 & 0 & 2 & 2 \\ 4 & 1 & 3 & 1 \end{bmatrix} \xrightarrow{R_2 = R_1 - \frac{1}{2}R_2} \begin{bmatrix} 1 & 1 & 0 & -2 \\ 0 & 1 & -1 & -3 \\ 4 & 1 & 3 & 1 \end{bmatrix}$

$R_3 = R_3 - 4R_1 = \begin{bmatrix} 1 & 1 & 0 & -2 \\ 0 & 1 & -1 & -3 \\ 0 & -3 & 3 & 9 \end{bmatrix}$

$R_3 = 3R_2 + R_3 \rightarrow \begin{bmatrix} 1 & 1 & 0 & -2 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$r(A) = 2, \quad n(A) = 4-2 = 2$

(3) Find the rank and nullity of D, given:

$D = \begin{bmatrix} 1 & 3 \\ 0 & -2 \\ 5 & -1 \\ -2 & 3 \end{bmatrix} \xrightarrow{R_3 = R_3 - 5R_1} \begin{bmatrix} 1 & 3 \\ 0 & -2 \\ 0 & -16 \\ -2 & 3 \end{bmatrix}$

$\xrightarrow{R_4 = R_4 - (2R_1)} \begin{bmatrix} 1 & 3 \\ 0 & -2 \\ 0 & -16 \\ 0 & 9 \end{bmatrix} \xrightarrow{R_3 = R_3 + 8R_2} \begin{bmatrix} 1 & 3 \\ 0 & -2 \\ 0 & 0 \\ 0 & 9 \end{bmatrix}$

$\xrightarrow{R_4 = R_4 + \frac{9}{2}R_2} \begin{bmatrix} 1 & 3 \\ 0 & -2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ ~~REF~~

$r(D) = 2, \quad n(D) = 2-2 = 0$ Ans

5th May, 2024

Topics

- Echelon reduction for inverse matrix and System of linear equation
- Eigen values and eigen vectors
- Similar matrices
- Canonical forms
- Quadratic form

Row Reduction and Echelon formation

Generally, a non-zero row or column in a matrix means a row or column that contains at least one entry that is non-zero

A leading ~~vector~~ entries in a row refers to the left most entry in that row

Definition

A rectangular matrix is in echelon form (or row echelon form) can be derived using the following steps:

(i) All non-zero rows are above any rows that are all zeros.

(ii) Each leading entry of a row is in a column to the right of the leading entry of the row above it

(iii) All entries in a column below a leading entry are zero.

Remark

In addition to the conditions listed above, if a matrix in echelon form satisfies the following additional conditions, then it is in a reduced echelon form (or reduced row echelon form)

(iv) The leading entry in each non-zero is 1

(v) Each leading one is the only non-zero entity in its column

Remark:

An echelon matrix or a reduced echelon matrix is one that is in echelon form or reduced echelon form respectively.

7th May, 2024

Definition: A pivot position in a matrix A is a location in A that corresponds to a leading 1 in the reduced echelon form of the matrix A.

A pivot column is a column of the matrix A that contains a pivot position

Example 1: Row reduce the matrix A below to its echelon form and locate the pivot columns of A

$$A = \begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix}$$

Solution

The top of the left most non-zero column is the pivot position. A non zero entry must be placed in this position. To make this happen, let us interchange rows 1 and 4 in the given matrix

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

Create zeros below the pivot 1 by adding multiples of the first row to the rows below it and obtain the next matrix

$$\begin{array}{l} R_1 \longrightarrow R_1 \\ R_2 \longrightarrow R_2 + R_1 \\ R_3 \longrightarrow R_3 + 2R_1 \\ R_4 \longrightarrow R_4 + 0R_1 \end{array} \begin{bmatrix} 1 & 4 & 5 & -4 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 5 & 10 & -15 & -15 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

$$\begin{array}{l} R_1 \to R_1 \\ R_2 \to R_2 \\ R_3 \to -\frac{5}{2}R_2 + R_1 \\ R_4 \to \frac{3}{2}R_2 + R_4 \end{array} \begin{bmatrix} 1 & 4 & 5 & -4 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 \end{bmatrix}$$

There is no way to create a leading entry in column 3, however if we interchange row 3 and row 4, we can produce a leading entry in column 4

$$\begin{array}{l} R_1 \to R_1 \\ R_2 \to R_2 \\ R_3 \to R_4 \\ R_4 \to R_3 \end{array} \begin{bmatrix} 1^{\to \text{pivot}} & 4 & 5 & -4 & -7 \\ 0 & 2^{\to \text{pivot}} & 4 & -6 & -6 \\ 0 & 0 & 0 & -5^{\to \text{pivot}} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \text{rank}$$

In this case, the matrix is already in echelon form and it is also clear that columns 1, 2 and 4 of the matrix A are the pivot columns and the actual pivots in this particular example are 1, 2 and -5

Remark: (i) The rank of a matrix is the number of non-zero rows in the echelon form of ~~the~~ a matrix. In this example above, the rank of the matrix is 3.

(ii) The nullity of a matrix is the number of zero rows in the echelon form of that matrix.

In the example above, the nullity of the matrix A is 1

Definition: The dimension of a matrix is the same addition of the rank and the nullity

$$\text{Dim}(A) = \text{rank}(A) + \text{nullity}(A)$$
$$= 3 + 1 = 4$$

Example 2: Use the elementary row operations to transform the following matrix into echelon form and thereafter, into a reduced echelon form.

$$C = \begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -8 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix}$$

Step 1: Interchange row 3 for row 1

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 3 & -7 & 8 & -8 & 8 & 9 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

Step 2: Use R_1 to reduce other rows

$$\begin{array}{l} R_1 \to R_1 \\ R_2 \to R_2 - R_1 \\ R_3 \to R_3 + 0R_1 \end{array} \begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

Step 3: Use R_2 to reduce other rows

$$\begin{array}{l} R_1 \to R_1 \\ R_2 \to R_2 \\ R_3 \to R_3 - \frac{3}{2}R_2 \end{array} \begin{bmatrix} 3^{\to \text{pivot}} & -9 & 12 & -9 & 6 & 15 \\ 0 & 2^{\to \text{pivot}} & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & 1^{\to \text{pivot}} & 4 \end{bmatrix}$$

rank = 3, nullity = 0, dimension = 3

pivot = 3, 2 and 1

11th May, 2024

To obtain the reduce echelon form of the last example, we can begin with the rightmost pivot and working upwards and to the left to create zeros above each pivot.

If a pivot is not 1, make it is 1 by scaling operation.

In this particular example, the rightmost pivot is in row 3. We can create zeros above it, by adding suitable multiples of row 3 to row 2 and 1

$$\begin{array}{l} R_1 \to R_1 - 6R_3 \\ R_2 \to R_2 - 2R_3 \\ R_3 \to R_3 \end{array} \begin{bmatrix} 3 & -9 & 12 & -9 & 0 & -9 \\ 0 & 2 & -4 & 4 & 0 & -14 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

The next pivot is in Row 2, but we have to scale this pivot by dividing Row 2 by 2

$$\begin{array}{l} R_1 \to R_1 \\ R_2 \to \frac{1}{2}R_2 \\ R_3 \to R_3 \end{array} \begin{bmatrix} 3 & -9 & 12 & -9 & 0 & -9 \\ 0 & 1 & -2 & 2 & 0 & -9 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

$$\begin{array}{l} R_1 \to R_1 + 9R_2 \\ R_2 \to R_2 \\ R_3 \to R_3 \end{array} \begin{bmatrix} 3 & 0 & -6 & 9 & 0 & -72 \\ 0 & 1 & -2 & 2 & 0 & -3 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

Finally, we can scale row 1 by 3, by dividing by pivot 3, we shall then obtain;

$$\begin{bmatrix} 1 & 0 & -2 & 3 & 0 & -24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

This is the reduced form of the echelon matrix for the given original matrix.

20th May, 2024

Eigen Values of Eigen Vectors

Let A be a $n \times n$ matrix, if there exist a real value λ and a non zero $n \times 1$ matrix, satisfying

$Ax = \lambda x$, then we refer to λ as an eigen value of the matrix A and X as an eigen vectors of A corresponding to λ.

Example

If $A = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}$

We can observe that if we multiply

$$\begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

Hence 3 is an eigen value of the matrix A and $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ is an eigen vector of A corresponding to 3.

Finding All Eigenvalues of a Matrix

Consider the equation $(A - \lambda I)X = 0$, where I is the $I = n \times n$, identity matrix, by introducing $B = A - \lambda I$, we can rewrite the above equation as $BX = 0$

This then means that the above equation is an equation above X

Then if the determinant of $(B) \neq 0$. The above equation has a unique solution $X = 0$. However this is not what we need.

The goal is to find an eigen value of the matrix A and an eigen vector X of A which needs to be a non zero matrix.

Therefore, we must choose λ appropriately, so that $\det(B) = 0$ i.e

$\det(A - \lambda I) = 0$

This provide us a way to find all the eigen values of the matrix A

$(A - \lambda I)x = 0$

There ~~X~~ cannot be 0. $x = 0$ if λ is 0

Example

Consider $A = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}$

$B = A - \lambda I = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ because the 2x2 matrix

$= \begin{bmatrix} 1-\lambda & -1 \\ 2 & 4-\lambda \end{bmatrix}$

$\det(B) = \det(A - \lambda I) = 0$

$\begin{bmatrix} 1-\lambda & -1 \\ 2 & 4-\lambda \end{bmatrix} = 0$

$(1-\lambda)(4-\lambda) + 2 = 0$

$\lambda^2 - 5\lambda - 6 = 0$

$\lambda_1 = 3, \lambda_2 = 2$

These are the eigen value of the matrix A.

Remark:

In general $\det(A - \lambda I)$ is always a polynomial function of λ.

We refer to the function as the characteristic polynomial of the matrix A.

For instance, in the previous example, the characteristic polynomial of A is

$[\lambda^2 - 5\lambda + 6]$

The Equation: $[\det[A - \lambda I] = 0]$ is called The characteristic Equation

Summary

* We formulate $A - \lambda I = 0$
* Find determinate
* get ~~a~~ polynomial

Finding All Eigen Vectors of the Matrix A

Let λ be an eigen value of the matrix A, then the equation $(A - \lambda I)x = 0$ has infinitely many solutions x because the $\det(A - \lambda I) = 0$.

It will be shown that all those solutions x constitute a vector space which is denoted as eigen space λ.

We know that $\lambda_1 = 3$ and $\lambda_2 = 2$ are the eigen values of the matrix A.

To obtain the eigen vector of A corresponding to $\lambda_1 = 3$, we want to find x which is in $x \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ satisfying the equation

$(A - \lambda_1 I)x = 0$ i.e

$\begin{bmatrix} 1-3 & -1 \\ 2 & 4-3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

row reduction

$\begin{bmatrix} -2 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

i.e $-2x_1, x_2 = 0$ is a solution

To the above system

The set of such vectors can be represented in parametric form as

$x_1 = t, x_2 = -2t$ for any $t \in \mathbb{R}$

Thus $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} t \\ -2t \end{bmatrix}$

$t = \begin{bmatrix} t \\ -2 \end{bmatrix}$

Therefore $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ is an eigen

vector of corresponding to $\lambda = 3$

Similarly, $\lambda_2 = 2$

$$X = \begin{bmatrix} t \\ -t \end{bmatrix} = t \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$\Rightarrow \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is an eigen vector of A corresponding to $\lambda_2 = 2$

20th June, 2024

Example: Consider the matrix

$$A = \begin{bmatrix} 4 & 6 & 0 \\ -3 & -5 & 0 \\ -3 & -6 & 1 \end{bmatrix}$$

prove that the matrix A has only two usable eigen values. Show further that the only eigen vectors of A corresponding to those eigen values is $\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$ and eigen space (λ_1) with

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2u \\ -u \\ v \end{bmatrix} \text{ and } (u, v) \in \mathbb{R}^2$$

Solution: The characteristic equation is

$\det(A - \lambda I) = 0$

$$|A - \lambda I| = \begin{vmatrix} 4-\lambda & 6 & 0 \\ -3 & -5-\lambda & 0 \\ -3 & -6 & 1-\lambda \end{vmatrix} = 0$$

$$= (1-\lambda) \begin{vmatrix} 4-\lambda & 6 \\ -3 & -5-\lambda \end{vmatrix} = 0$$

$(1-\lambda)\left[(4-\lambda)(-5-\lambda) + 18\right] = 0$

$(\lambda-1)\left[(4-\lambda)(5+\lambda) - 18\right] = 0$

$(\lambda-1)\left[20 + 4\lambda - 5\lambda - \lambda^2 - 18\right] = 0$

$(\lambda-1)(2 - \lambda - \lambda^2) = 0$

$(1-\lambda)(\lambda^2 + \lambda - 2) = 0$

$(1-\lambda)^2(\lambda + 2) = 0$

$\lambda_1 = 1, \lambda_2 = -2$

for $\lambda_1 = 1$

$(A - \lambda_1 I)X = 0$

$$\begin{bmatrix} 4-1 & 6 & 0 \\ -3 & -5-1 & 0 \\ 3 & -6 & 1-1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 6 & 0 \\ -3 & -6 & 0 \\ -3 & -6 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

row reduction

$$\begin{bmatrix} 3 & 6 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow 3x_1 + 6x_2 = 0$

$3x_1 + 6x_2 + 0x_3 = 0$

$x_1 = 2u, \quad x_2 = -u, \quad x_3 = v$

$(u, v) \in \mathbb{R}$

This is a vector space denoted by eigen path (λ_1)

$\lambda_2 = -2$

$(A - \lambda_1 I)X = 0$

$$\begin{bmatrix} 4+2 & 6 & 0 \\ -3 & -3 & 0 \\ -3 & -6 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

row reduction

$$\begin{bmatrix} -1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-x_1 - x_2 = 0 \Rightarrow x_2 = -x_1$

$x_2 - x_3 = 0 \quad x_3 = x_3 = -2,$

$\therefore x_1 = -t, \ x_2 = t, \ x_3 = t$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -t \\ t \\ t \end{bmatrix} = t\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \therefore \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$