MTH 202 NOTE

2018/19 SESSION

NOTE CREDIT .;CHRISTIANA

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27/08/2019 VECTOR THEORY

VECTOR CALCULUS;

Combination of i) vector theories
ii) differential differentiation & integration

$\forall \vec{a}, \vec{b}, \vec{c}, \ldots, \vec{v} \in \mathbb{V}(\mathbb{R}^{n+1}) \, n \in \mathbb{N}$

1. Scalar product $\vec{a} \cdot \vec{b} = \alpha \in \mathbb{R}$
2. Vector product $\vec{a} \times \vec{b} = \vec{v} \in \mathbb{V}$
3. Scalar triplet $\vec{a} \cdot (\vec{b} \times \vec{c}) = \det(\vec{a}, \vec{b}, \vec{c}) = \beta \in \mathbb{R}$
4. Vector triplet $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$
$= \alpha\vec{b} + \beta\vec{c}$

NABLA OPERATOR ∇

$\nabla = \vec{K}_n \frac{\partial}{\partial x_n} \quad n = 1, 2, 3, \ldots, m$

$= \vec{k}_1 \frac{\partial}{\partial x_1} + \vec{k}_2 \frac{\partial}{\partial x_2} + \vec{k}_3 \frac{\partial}{\partial x_3} + \ldots + \vec{k}_n \frac{\partial}{\partial x_n}$

$m = 3, \; \nabla = \vec{k}_1 \frac{\partial}{\partial x_1} + \vec{k}_2 \frac{\partial}{\partial x_2} + \vec{k}_3 \frac{\partial}{\partial x_3}$

$= \vec{i} \frac{\partial}{\partial x} + \vec{j} \frac{\partial}{\partial y} + \vec{k} \frac{\partial}{\partial z}$

1. $\nabla \cdot \vec{b}$ degree of $\vec{b}$
2. $\nabla \times \vec{b}$ rotor/spin/curl/torque of $\vec{b}$
3. $\nabla \cdot (\vec{b} \times \vec{c}) \equiv \nabla \cdot \vec{v}$ divergence of $\vec{v}$
4. $\nabla \times (\vec{b} \times \vec{c}) \equiv \nabla \times \vec{v}$ rotor/curl of $\vec{v}$
5. $\nabla \phi (\vec{x})$ gradient of scalar function

$\nabla \phi$ = vector gradient

$\nabla\vec{b} = \nabla \otimes \vec{b}$ = tensor gradient.

28/08/19.

FIELD

$a = a_1\vec{k_1} + a_2\vec{k_2} + a_3\vec{k_3}$

a_1, a_2, a_3 - component

k_1, k_2, k_3 - basis.

Now, we view the vectors and scalars not just as relating to a fixed point in space or region but as being $\vec{a} = \vec{a}(x_i)$ $i = 1, 2 \ldots n$, $\phi = \phi(x_i)$ $i = 1, 2, \ldots, n$ i.e not limited to the three dimensional space continuously defined over a region or space which we call R^n, $n \in N$. We write $\phi(x_i) \in C^{\infty}(R^n)$ and $a(x_i) \in C^{\infty}(R^n)$ where $i = 1, 2, \ldots, n$.

We have written that ϕ is in the space of continuously differentiable set or space

$\phi \in C^0(R)$ continuous.

$\phi \in C^1(R^n)$ continuous & differentiable once

$\phi \in C^2(R^n)$ continuous & differentiable twice

In ∞ case, we say ϕ and $\vec{a}$ are smooth in the region in the space R^n.

SCALAR FIELD

- Gravitational field in space.
- Pressure in every point of a fluid (gas or liquid / plasma).
- Electrostatic potential in an electric region. In this case, electric field constitute the vector field of that system.
- Length of a vector

VECTOR FIELD

Velocity of particles in that field medium.

Electric potential vector field.

Gravitational field between particles

Potential field

$\forall \phi : \vec{a} = \nabla \phi$.

With these continuous varying quantity or concept of field comes the imperative of their derivatives or differentiation and integration over the space where they are defined $\phi(\vec{x})$, $\vec{a}(\vec{x})$. Hence,

VECTOR DIFFERENTIAL OPERATOR

NABLA ∇

$$\nabla \equiv \vec{e}^{\,i} \frac{\partial}{\partial q^i}, \quad i = 1, 2, \ldots, n$$

where $\vec{e}^{\,i}$ contrabasis (inverse vector-basis)

q^i any coordinate in the space of consideration

$$\nabla = k_i \frac{\partial}{\partial x_i}, \quad i = 1, 2, 3 \qquad \nabla = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}$$

Example: Find the gradient of a scalar function $\phi(x,y,z) = x^3y^3z$.

$\nabla\phi$; $(\vec{i}\,\partial/\partial x + \vec{j}\,\partial/\partial y + \vec{k}\,\partial/\partial z)(x^2y^3z)$

$= 2xy^3z\vec{i} + 3x^2y^2z\vec{j} + x^2y^3\vec{k}$ ⇒ gradient of ϕ

The gradient of a scalar field is vector.

$\nabla\phi \to$ gradient

$\nabla\cdot\vec{a} \to$ divergence.

$\nabla\cdot\vec{a} = 0$ the field $\vec{a}$ is divergence free or solenoided.

$\nabla\times\vec{a} \longrightarrow$ rotational / curl / spin / rotor

$\nabla\times\vec{a} = 0$ we say $\vec{a}$ is rotation free or irrotational.

$\nabla\otimes\vec{a} \to$ tensor gradient $\varepsilon = \frac{1}{2}(\nabla\vec{a} + \nabla\vec{a})$ strain,

22/08/19. $\vec{a}: \exists\phi(\vec{x}): \vec{a} = \nabla\phi$. In this case, we say that the field is potential. A field $\vec{v}$ is

i) potential if $\exists\phi(\vec{x}); \vec{v} = \nabla\phi$, the field $\vec{v}$ is potential

ii) irrotational if $\nabla\times\vec{v} = 0$ ie $\vec{v}$ is spin-free

iii) solenoidal if $\nabla\cdot\vec{v} = 0$ ie $\vec{v}$ is divergence-free

$\nabla\cdot(\phi\vec{v}) = \nabla\phi\cdot\vec{v} + \phi\,\nabla\cdot\vec{v}$ — taking the div of a vector, you get a scalar.

$\nabla\times(\phi\vec{v}) = \nabla\phi\times\vec{v} + \phi\,\nabla\times\vec{v}$

Gradient field ≡ Potential field < irrotational field; conservative field (independent of path taken)

* A conservative field of necessity is irrotational
* For an irrotational field to lead to conservative field to lead to conservative field, the domain must be connected.

$\int \vec{a}\cdot d\vec{r} = \int \nabla\phi\cdot d\vec{r} = \int d\phi$

$= \phi\big|_A^B = \phi(B) - \phi(A)$

irrotationality $\vec{a}$; $\nabla\times\vec{a} = 0$

$\nabla\times\nabla\times\vec{a} = 0$

Behavior of Gradient Field ($\nabla\phi$, grad ϕ)

i. Defines directional derivative: $\frac{\partial\phi}{\partial l} = \nabla\phi\cdot\vec{i}$

ii. Normal to potential surface: $\phi(\vec{r}) = c$ $c\in\mathbb{R}$

iii. Nabla is parallel to position vector $\vec{r}$: $\nabla\times\vec{r} = 0$

Proof:

1. Let $\phi(x(t), y(t), z(t)) = c$ (constant) (Gradient field). (taking derivative of both sides)

$0 = \frac{dc}{dt} = \frac{d\phi}{dt} = \frac{\partial\phi}{\partial x}\frac{dx}{dt} + \frac{\partial\phi}{\partial y}\frac{dy}{dt} + \frac{\partial\phi}{\partial z}\frac{dz}{dt}$ (ϕ is dependent on c but not vice versa)

$= \left(\frac{\partial\phi}{\partial x}\vec{i} + \frac{\partial\phi}{\partial y}\vec{j} + \frac{\partial\phi}{\partial z}\vec{k}\right)\cdot\left(\frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j} + \frac{dz}{dt}\vec{k}\right)$

$= (\nabla\phi)\cdot(\vec{r})$

$= \nabla\phi\cdot\vec{r} = 0 \Rightarrow \nabla\phi \perp \vec{r} \Rightarrow \nabla\phi \parallel \vec{r}$

$\frac{d\phi}{dl} = \nabla\phi\cdot\vec{i}$

$\vec{a}\times\vec{a} = 0$

$\vec{a}\times\vec{v} = 0 \Leftarrow \vec{v} = \alpha\vec{a}$ ($\vec{v} = \vec{a}$)

$\vec{a}\times(\alpha\vec{a})$

$\alpha(\vec{a}\times\vec{a}) = 0$

$\nabla\times\vec{r} = 0$

$$\nabla\times\vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = \vec{i}\left(\frac{\partial z}{\partial y} - \frac{\partial y}{\partial z}\right) + \vec{j}\left(\frac{\partial x}{\partial z} - \frac{\partial z}{\partial x}\right) + \vec{k}\left(\frac{\partial y}{\partial x} - \frac{\partial x}{\partial y}\right)$$

$= \vec{i}0 + \vec{j}0 + \vec{k}0 = \vec{0}$

They all go to zero because (x, y, z) are all independent of each other, hence their derivative goes to zero.

$\nabla \equiv \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}$

$\vec{r} = \vec{i}x + \vec{j}y + \vec{k}z$.

$\nabla \times \vec{r} = (\vec{k_n}\frac{\partial}{\partial x_n}) \times (\vec{k_m} x_m)$

$= \frac{\partial x_m}{\partial x_n}\vec{k_n} \times \vec{k_m} = \delta_{mn}\, \epsilon_{nmp}\vec{k_p}$

$\delta_m = \begin{cases} 1 & m = n \\ 0 & m \neq n \end{cases}$ $\epsilon_{nmp} = \begin{cases} 1 \text{ if } n \neq m \neq p \neq n & P(nmp) \text{ is even} \\ 0 \text{ if any } n, m, p \text{ equal} \\ -1 \text{ if } n \neq m \neq p \neq n, & P(nmp) \text{ is odd} \end{cases}$

30/08/19. Example

i) Proof that $\nabla\phi$ is a vector perpendicular to the surface (level curve) $\phi(x, y, z) = c$; $c \in \mathbb{R}$

ii) Find a unit vector normal to the surface $2x^2 + 4yz - 5z^2 = 3$ at point $P(3, -1, 2)$

Solution

Given surface $\phi(x, y, z) = 3$

$\nabla\phi$; $\hat{n} = \frac{\nabla\phi}{|\nabla\phi|}$, $\phi \equiv 2x^2 + 4yz - 5z^2$

$\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$

$= 4x\vec{i} + 4z\vec{j} - (10z - 4y)\vec{k}$

$|\nabla\phi| = (\nabla\phi\cdot\nabla\phi)^{1/2} = [16x^2 + 16z^2 + (4y - 10z)^2]^{1/2} = 28$

$= (144 + 64 + 576)^{1/2} = \sqrt{784}$

$\Rightarrow \hat{n} = \frac{1}{28}\nabla\phi = \frac{1}{28}[4x\vec{i} + 4z\vec{j} + (4y - 10z)\vec{k}]$

Example: Find the rate of change wrt distance s of the function $\phi = 2x^2y + yz$ at point $P(-2, 2, -4)$ in the direction $\vec{v} = \vec{i} + 2\vec{j} + 3\vec{k}$. Estimate its maximum value.

Solution:

i) Rate of change of $\phi \to$ grad ϕ : $\nabla\phi$.

ii) $\frac{d\phi}{ds} \to \nabla\phi\cdot\hat{n} \Rightarrow \frac{d\phi}{ds} = \nabla\phi\cdot\hat{n} = \nabla\phi\cdot\hat{v}$

$d\phi/ds = \nabla\phi\cdot\hat{n}$; $\hat{v} = \vec{v}/|\vec{v}|$.

$\nabla\phi = 4xy\,i + (2x^2 + z)\vec{j} + y\vec{k}$

$|\nabla\phi| = (16^2 + 4^2 + 2^2)^{1/2} = \sqrt{276}$

$\Rightarrow |\vec{v}| = \sqrt{14}$ $\hat{v} = 1/14(\vec{i} + 2\vec{j} + 3\vec{k})$

$\Rightarrow \frac{d\phi}{ds} = \frac{1}{\sqrt{14}}(\vec{i} + 2\vec{j} + 3\vec{k})\cdot[4xy\vec{i} + (2x^2 + z)\vec{j} + y\vec{k}]$

$= 1/\sqrt{14}\,[4xy + 2(2x^2 + z) + 3y]_{(-2, 2, -4)}$

$= 1/\sqrt{14}\,(-16 + 8 + 6) = -2/\sqrt{14}$

$\left|\frac{d\phi}{ds}\right| = |\nabla\phi\cdot\hat{n}| = |\nabla\phi\cdot\hat{v}| = ||\nabla\phi||\hat{v}|\cos\theta| = ||\nabla\phi|\cos\theta|$

$|\cos\theta| = 1$ $\left|\frac{d\phi}{ds}\right| \to |\nabla\phi|$; $\cos\theta = 1$ $\theta = 0$

Properties of gradient ∇ and divergence ∇

$\forall \vec{a}, \vec{b} \in V(\mathbb{R}^n)$: $k, c \in \mathbb{R}$; $\phi, \phi \in \mathbb{R}^n$

i. $\nabla(k\phi) = k(\nabla\phi)$

ii. $\nabla(k\phi + c\phi) = k\nabla\phi + c\nabla\phi$

iii. $\nabla\phi\phi = \phi\nabla\phi + \phi\nabla\phi$

iv. $\nabla \phi^n = n\phi^{n-1} \nabla \phi$

v. $\nabla \frac{\phi}{\phi} = \frac{1}{\phi^2} (\phi \nabla \phi + \phi \nabla \phi)$

vi. $\nabla^2 (\phi \phi) = \phi \nabla^2 \phi + 2 \nabla \phi \nabla \phi + \phi \nabla^2 \phi$

i. $\nabla \cdot (k\vec{a}) = k \nabla \cdot \vec{a}$

ii. $\nabla \cdot (k\vec{a} + c\vec{b}) = k\nabla \cdot \vec{a} + c \nabla \cdot \vec{b}$

iii. $\nabla \cdot \phi \vec{a} = \phi \nabla \cdot \vec{a} + \vec{a} \cdot \nabla \phi$

iv. $\nabla \cdot (\phi \nabla \phi) = \phi \nabla^2 \phi + \nabla \phi \cdot \nabla \phi$

v. $\nabla \cdot (\phi \nabla \phi) - \nabla \cdot (\phi \nabla \phi) = \phi \Delta \phi + \phi \Delta \phi$

$$\nabla^2 = \nabla \cdot \nabla = \Delta \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

Example: If $\vec{a} = \vec{i} yz + \vec{j} xz + \vec{k} xy$, find

i) $\nabla \cdot \vec{a}$ ii) $\nabla \times \vec{a}$

Solution

i) $\nabla \cdot \vec{a} = \vec{i}0 + \vec{j}0 + \vec{k}0 = \vec{0} \Rightarrow \vec{a}$ is solenoidal.

ii. $\nabla \times \vec{a} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & xz & xy \end{vmatrix}$

$= (x-x)\vec{i} + (y-y)\vec{j} + (z-z)\vec{k} = 0$

$\Rightarrow \nabla \times \vec{a} = 0$

Assignment

Evaluate $r = |\vec{r}| = \sqrt{x^2+y^2+z^2}$ $\vec{r} \equiv x\vec{i} + y\vec{j} + z\vec{k}$

i) ∇r ii) $\nabla \frac{1}{r}$ iii) $\nabla \ln r$ iv) $\nabla \ln \frac{1}{r}$ v) $\nabla^2 r$

vi) $\nabla^2 1/r$ vii) $\nabla^2 \ln r$ viii) $\nabla^2 \ln \frac{1}{r}$

Show that $\nabla^2 r^{n+1} = (n+2)(n+1) r^{n-1}$, $n \in \mathbb{Z}$ and discuss $n = -2$, $n = -1$, $n = 0$, $n = 1$, $n = 2$, $n = 3$, $n =$

Show that $r^\alpha \vec{r}$ is an irrotational vector $\forall \alpha$; but is solenoidal only if $\alpha = -3$.

Example $r = \sqrt{x^2+y^2+z^2}$

i. $\frac{\partial r}{\partial x} = \frac{\partial}{\partial x} (x^2+y^2+z^2)^{1/2} = \frac{1}{2} (x^2+y^2+z^2)^{1/2} \frac{\partial}{\partial x} (x^2)$

$= x/r$.

ii. $\frac{\partial^2}{\partial x^2} (\ln r) = \frac{r^2 - 2x^2}{r^4}$

$\nabla r = \frac{x}{r}\vec{i} + \frac{y}{r}\vec{j} + \frac{z}{r}\vec{k}$

$= \frac{1}{r} (x\vec{i} + y\vec{j} + z\vec{k})$

$= \frac{1}{r}\vec{r} = \nabla r = \hat{r}$

3/09/2019

LAPLACIAN OPERATOR

Let Φ be a scalar field defined in a region R of the Euclidean space described by the rectangular cartesian coordinate system. If we impose a condition of Φ and is differentiable of second order, then the Laplacian of Φ denoted as $\nabla^2\Phi$ is defined as

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$$

The operator denoted as ∇^2 is called the Laplacian Operator which is also denoted as Δ and it is defined as

$$\nabla^2 = \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

Clearly:

$$\nabla^2 = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\cdot\left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)$$

i.e. $\nabla^2 = \nabla\cdot\nabla$

in ϕ, we have

$$\nabla^2\phi = \Delta\phi = \nabla\cdot(\nabla\phi)$$

Remark;

- Laplacian of a scalar field is a scalar field
- Laplacian of a vector field is a vector field
- Divergence of gradient of a field is not the same as gradient of divergence of a field. In fact, divergence of gradient of a field is the Laplacian of the field.

Example:

Let $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ be the position vector of an arbitrary point in a region R of the Euclidean space described by the rectangular cartesian coordinate system. Let $r = |\vec{r}|$ be the magnitude of $\vec{r}$. Show that;

$$\nabla^2\left(\frac{1}{r}\right) = 0$$

$r = \sqrt{x^2+y^2+z^2}$ (Taking inverse of both sides)

$$\frac{1}{r} = (x^2+y^2+z^2)^{-1/2}$$

$$\nabla^2\left(\frac{1}{r}\right) = \frac{\partial^2}{\partial x^2}\left(\frac{1}{r}\right) + \frac{\partial^2}{\partial y^2}\left(\frac{1}{r}\right) + \frac{\partial^2}{\partial z^2}\left(\frac{1}{r}\right)$$

$$\frac{\partial^2}{\partial x^2}\left(\frac{1}{r}\right) = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\left[\frac{1}{r}\right]\right) = \frac{\partial}{\partial x}\left[\frac{-x}{r^3}\right]$$

$$= \frac{\partial}{\partial x}\left(-xr^{-3}\right) = \frac{-1}{r^3} - x\times\frac{(-x)}{r^5}\times 3$$

by chain rule

$$\left\{\frac{\partial}{\partial x}r^{-3} = \frac{dr^{-3}}{dr}\cdot\frac{\partial r'}{\partial x} \rightarrow \left(\frac{\partial}{\partial x}\left(x^2+y^2+z^2\right)^{1/2}\right)\right\}$$

$$\frac{\partial^2}{\partial x^2}\left(\frac{1}{r}\right) = -\frac{1}{r^3} + \frac{3x^2}{r^5}$$

Similarly

$$\frac{\partial^2}{\partial y^2}\left(\frac{1}{r}\right) = -\frac{1}{r^3} + \frac{3y^2}{r^5}.$$

$$\frac{\partial^2}{\partial z^2}\left(\frac{1}{r}\right) = -\frac{1}{r^3} + \frac{3z^2}{r^5}.$$

$$\nabla^2\left(\frac{1}{r}\right) = -\frac{1}{r^3} + \frac{3x^2}{r^5} - \frac{1}{r^3} + \frac{3y^2}{r^5} - \frac{1}{r^3} + \frac{3z^2}{r^5}.$$

$$= \frac{-3}{r^3} + \frac{3}{r^5}\left(x^2 + y^2 + z^2\right)$$

$$= \frac{-3}{r^3} + \frac{3r^2}{r^5} = -\frac{3}{r^3} + \frac{3}{r^3} = 0.$$

Theorem - Let $\vec{r}$ be given as $x\vec{i} + y\vec{j} + z\vec{k}$ be position of an arbitrary point in a region R of the Euclidean space. Suppose f is a scalar function defined in a subset D of the set of real numbers whose value at $r \in D$ is $f(r)$, with $r = |\vec{r}|$; then if f has second order derivative

1. $\nabla f(r) = \frac{\vec{r}}{r}\frac{df(r)}{dr}$

2. $\nabla^2 f(r) = \frac{d^2 f(r)}{dr^2} + \frac{2}{r}\frac{df(r)}{dr}$

Using the same example;

$$\nabla^2\left(\frac{1}{r}\right) = \frac{d^2}{dr^2}\left(\frac{1}{r}\right) + \frac{2}{r}\frac{d}{dr}\left(\frac{1}{r}\right)$$

$$= \frac{2}{r^3} + \frac{2}{r} \times \frac{-1}{r^2} = \frac{2}{r^3} - \frac{2}{r^3} = 0.$$

Proof:-

By definition;

$$\nabla f(r) = \vec{i}\frac{\partial}{\partial x}f(r) + \vec{j}\frac{\partial}{\partial y}f(r) + \vec{k}\frac{\partial}{\partial z}f(r).$$

Recall; By chain rule;

$$\frac{\partial f(r)}{\partial x} = \frac{df(r)}{dr}\frac{\partial r}{\partial x} = \frac{x}{r}\frac{df(r)}{dr}$$

Similarly (By symmetry)

$$\frac{\partial f(r)}{\partial y} = \frac{df(r)}{dr}\frac{\partial r}{\partial y} = \frac{y}{r}\frac{df(r)}{dr}$$

$$\frac{\partial f(r)}{\partial z} = \frac{z}{r}\frac{df(r)}{dr}$$

$$\nabla f(r) = \vec{i}\frac{x}{r}\frac{df(r)}{dr} + \vec{j}\frac{y}{r}\frac{df(r)}{dr} + \vec{k}\frac{z}{r}\frac{df(r)}{dr}$$

$$= (x\vec{i} + y\vec{j} + z\vec{k}) \times \frac{1}{r}\frac{df(r)}{dr}$$

$$\therefore \nabla f(r) = \frac{\vec{r}}{r}\frac{df(r)}{dr}.$$

4/09/2019. Proof

i. $\nabla^2 f(r) = \frac{d^2}{dr^2} f(r) + \frac{2}{r}\frac{d}{dr} f(r)$.

Recall;

$$\nabla\cdot\vec{r} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 3$$

$$\nabla\cdot(\phi\vec{V}) = \phi\,\nabla\cdot\vec{V} + \vec{V}\cdot\nabla\phi$$

$$\nabla^2 f(r) = \nabla\cdot(\nabla f(r))$$

$$= \nabla\cdot\left(\frac{\vec{r}}{r}\frac{df(r)}{r}\right)$$

$$= \frac{1}{r}\frac{df(r)}{dr}\nabla\cdot\vec{r} + \vec{r}\cdot\nabla\left(\frac{1}{r}\frac{df(r)}{dr}\right).$$

$$* \quad g(r) = \frac{1}{r}\frac{d}{dr}f(r)$$

$$\nabla g(r) = \frac{\vec{r}}{r}\frac{dg(r)}{dr} = \frac{\vec{r}}{r}\frac{d}{dr}\left[\frac{1}{r}\frac{df(r)}{dr}\right] \quad *$$

$$= \frac{3}{r}\frac{df(r)}{dr} + \vec{r}\cdot\frac{\vec{r}}{r}\frac{d}{dr}\left[\frac{1}{r}\frac{df(r)}{dr}\right]$$

$$= \frac{3}{r}\frac{df(r)}{dr} + \frac{r^2}{r}\left[\frac{-1}{r^2}\frac{df(r)}{dr} + \frac{1}{r}\frac{d^2f(r)}{dr^2}\right]$$

$$= \frac{3}{r}\frac{df(r)}{dr} - \frac{1}{r}\frac{df(r)}{dr} + \frac{d^2f(r)}{dr^2}$$

$$= \frac{2}{r}\frac{df(r)}{dr} + \frac{d^2f(r)}{dr^2} \quad \therefore \nabla^2 f(r) = \frac{2}{r}\frac{df(r)}{dr} + \frac{d^2f}{dr^2}$$

$$\nabla^2 f(r) = \frac{d^2f(r)}{dr^2} + \frac{2}{r}\frac{df(r)}{dr}.$$

Exercise :- Evaluate the following.

i) $\nabla^2 \ln r$ ii) $\nabla^2 r^{n+1}$ iii) $\nabla^2\left(\nabla\cdot\left(\frac{\vec{r}}{r^2}\right)\right)$

CURVILINEAR COORDINATE SYSTEM

Let $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ be the position vector of a point A in a region of the Euclidean space where (x, y, z) describes the rectangular coordinates. Suppose x, y, z are functions of the independent variables u, v, w in the sense;

$x = x(u, v, w)$, $y = y(u, v, w)$, $z = z(u, v, w)$

This is referred to as transformation of coordinates an (u, v, w) the curvilinear coordinate system or Material coordinate system.

It is possible to express u, v, w uniquely in terms of x, y and z provided the Jacobian of transformation of x, y and z with respect to u, v and w is not equal to zero. With this, we can write

$u = u(x, y, z)$ $v = v(x, y, z)$ and $w = w(x, y, z)$

Recall that Jacobian of transformation of x, y, z with respect to u, v, w denoted as $\frac{\partial(x, y, z)}{\partial(u, v, w)}$ or $J\left(\frac{x, y, z}{u, v, w}\right)$

defined as

$$J\left(\frac{x,y,z}{u,v,w}\right) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \\ \frac{\partial x}{\partial w} & \frac{\partial y}{\partial w} & \frac{\partial z}{\partial w} \end{vmatrix}$$

Recall; $\vec{a}\cdot(\vec{b}\times\vec{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

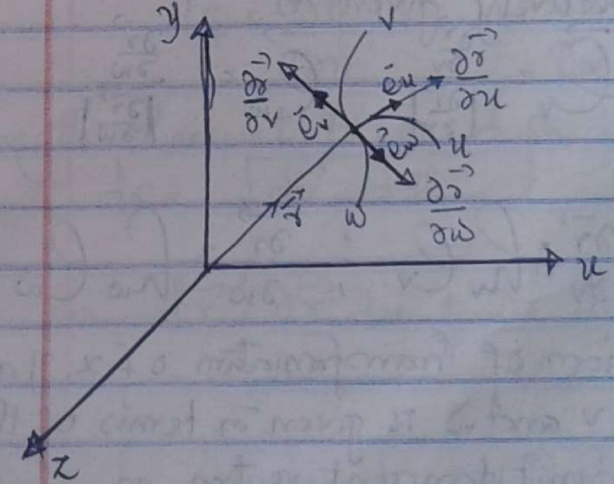
$$= \left(\vec{i}\frac{\partial x}{\partial u} + \vec{j}\frac{\partial y}{\partial u} + \vec{k}\frac{\partial z}{\partial u}\right)\cdot\left(\left[\vec{i}\frac{\partial x}{\partial v} + \vec{j}\frac{\partial y}{\partial v} + \vec{k}\frac{\partial z}{\partial v}\right]\times\left[\vec{i}\frac{\partial x}{\partial w} + \vec{j}\frac{\partial y}{\partial w} + \vec{k}\frac{\partial z}{\partial w}\right]\right)$$

$$= \frac{\partial \vec{r}}{\partial u}\cdot\left(\frac{\partial \vec{r}}{\partial v}\times\frac{\partial \vec{r}}{\partial w}\right)$$

$$J\left(\frac{x,y,z}{u,v,w}\right) = \frac{\partial \vec{r}}{\partial u}\cdot\left(\frac{\partial \vec{r}}{\partial v}\times\frac{\partial \vec{r}}{\partial w}\right)$$

By preservating the orientation of the coordinate system, we must have

$$J\left(\frac{x,y,z}{u,v,w}\right) > 0$$



SCALE FACTORS AND UNIT TANGENT VECTORS

Recall $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$= \vec{r}(u,v,w)$

A tangent vector in the increasing u-direction is given by $\frac{\partial \vec{r}}{\partial u}$

A tangent vector in the increasing v-direction is $\frac{\partial \vec{r}}{\partial v}$

A tangent vector in the increasing w-direction is $\frac{\partial \vec{r}}{\partial w}$.

The magnitude of these tangent vectors are called Scale factors denoted as h_u, h_v, h_w and defined respectively as

$$h_u = \left|\frac{\partial \vec{r}}{\partial u}\right| ; \quad h_v = \left|\frac{\partial \vec{r}}{\partial v}\right| ; \quad h_w = \left|\frac{\partial \vec{r}}{\partial w}\right|$$

Unit vectors $\vec{e}_u, \vec{e}_v$ and $\vec{e}_w$ along the u, v, w direction, are respectively given as:

$$\vec{e}_u = \frac{\frac{\partial \vec{r}}{\partial u}}{\left|\frac{\partial \vec{r}}{\partial u}\right|} \quad \vec{e}_v = \frac{\frac{\partial \vec{r}}{\partial v}}{\left|\frac{\partial \vec{r}}{\partial v}\right|} \quad \vec{e}_w = \frac{\frac{\partial \vec{r}}{\partial w}}{\left|\frac{\partial \vec{r}}{\partial w}\right|}$$

clearly, we have;

$$\frac{\partial \vec{r}}{\partial u} = h_u e_u ; \quad \frac{\partial \vec{r}}{\partial v} = h_v e_v ; \quad \frac{\partial \vec{r}}{\partial w} = h_w e_w$$

The Jacobian of transformation of x, y and z with respect to u, v and w is given in terms of the scale factor and unit tangent vectors as

$$J\left(\frac{x, y, z}{u, v, w}\right) = h_u h_v h_w \, \vec{e}_u \cdot (\vec{e}_v \times \vec{e}_w)$$

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The unit vectors $\vec{e}_u$, $\vec{e}_v$ and $\vec{e}_w$ from a basis in the sense that given any vector $\vec{a}$; we can write

$$\vec{a} = a_u \vec{e}_u + a_v \vec{e}_v + a_w \vec{e}_w$$

where a_u, a_v and a_w are the components of $\vec{a}$ with respect to the system (u, v, w)

ARC LENGTH, AREA AND VOLUME ELEMENTS.

The differential of a field ϕ is denoted as $d\phi$

if $\phi = \phi(u, v, w)$

$$d\phi = \frac{\partial \phi}{\partial u} du + \frac{\partial \phi}{\partial v} dv + \frac{\partial \phi}{\partial w} dw.$$

Arc length element ds is defined as

$$ds = \sqrt{d\vec{r} \cdot d\vec{r}}$$

$$d\vec{r} = \frac{\partial \vec{r}}{\partial u} du + \frac{\partial \vec{r}}{\partial v} dv + \frac{\partial \vec{r}}{\partial w} dw$$

$$d\vec{r} = h_u \vec{e}_u du + h_v \vec{e}_v dv + h_w \vec{e}_w dw$$

* For Orthogonal Coordinate System:

$$\vec{e}_u \cdot \vec{e}_v = 0; \quad \vec{e}_v \cdot \vec{e}_w = 0 \text{ and } \vec{e}_u \cdot \vec{e}_w = 0$$

This means that the basic vectors $\vec{e}_u, \vec{e}_v$ & $\vec{e}_w$ are

Mutually orthogonal in an orthogonal Coordinate System.

Therefore, arc length elements in an orthogonal system is given by

$$ds = \sqrt{h_u^2 du^2 + h_v^2 dv^2 + h_w^2 dw^2}$$

Area Element

Area element Orthogonal to $\vec{e_u}$, $\vec{e_v}$ and $\vec{e_w}$ will be denoted by dA_u, dA_v and dA_w and defined as

$$dA_u = |h_v \vec{e_v} dv \times h_w \vec{e_w} dw| = h_v h_w \, dv \, dw$$

Similarly

$$dA_v = h_u h_w \, du \, dw; \quad dA_w = h_u h_v \, du \, dv$$

Volume Element

Volume element dV^* is defined in an Orthogonal system as

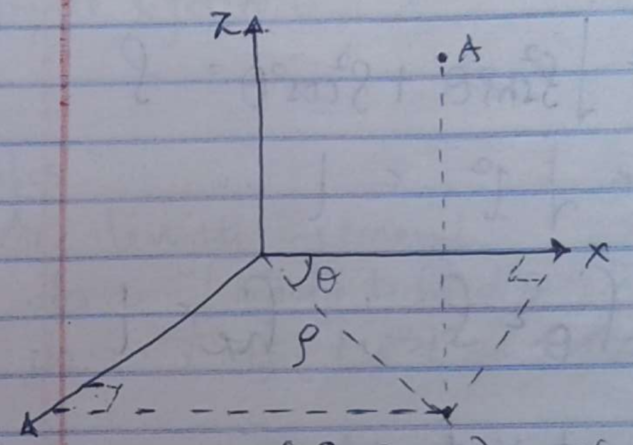
$$dV^* = |h_u \vec{e_u} du \cdot (h_v \vec{e_v} dv \times h_w \vec{e_w} dw)|$$

Simplified to

$$dV^* = h_u h_v h_w \, du \, dv \, dw$$

Examples Of Orthogonal Coordinate System.

A cylindrical coordinate system



From the diagram (ρ, θ, z) is the cylindrical coordinates. Their relations to x, y and z are given b

$$x = \rho\cos\theta; \quad y = \rho\sin\theta; \quad z = z$$

where $\rho \geq 0$; $0 \leq \theta < 2\pi$; $-\infty < z < \infty$

(i) $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$\therefore \vec{r} = \rho\cos\theta\vec{i} + \rho\sin\theta\vec{j} + z\vec{k}$

(ii) — Tangent vector

a. $\dfrac{\partial\vec{r}}{\partial\rho} = \cos\theta\vec{i} + \sin\theta\vec{j}$

b. $\dfrac{\partial\vec{r}}{\partial\theta} = -\rho\sin\theta\vec{i} + \rho\cos\theta\vec{j}$

c. $\dfrac{\partial\vec{r}}{\partial z} = \vec{k}$

(iii) Scale Factors :

(a) $h_\rho = \left|\frac{\partial \vec{r}}{\partial \rho}\right| = \sqrt{\cos^2\theta + \sin^2\theta} = 1$

(b) $h_\theta = \left|\frac{\partial \vec{r}}{\partial \theta}\right| = \sqrt{\rho^2\sin^2\theta + \rho^2\cos^2\theta} = \rho$

(c) $h_z = \left|\frac{\partial \vec{r}}{\partial z}\right| = \sqrt{1^2} = 1$.

$h_\rho = 1 \;;\; h_\theta = \rho \;;\; h_z = 1$.

(iv) Unit tangent Vectors :

(a) $\vec{e}_\rho = \frac{\frac{\partial \vec{r}}{\partial \rho}}{\left|\frac{\partial \vec{r}}{\partial \rho}\right|} = \frac{\cos\theta\,\vec{i} + \sin\theta\,\vec{j}}{1} = \cos\theta\,\vec{i} + \sin\theta\,\vec{j}$

(b) $\vec{e}_\theta = \frac{\frac{\partial \vec{r}}{\partial \theta}}{\left|\frac{\partial \vec{r}}{\partial \theta}\right|} = \frac{-\rho\sin\theta\,\vec{i} + \rho\cos\theta\,\vec{j}}{\rho} = -\sin\theta\,\vec{i} + \cos\theta\,\vec{j}$

(c) $\vec{e}_z = \frac{\frac{\partial \vec{r}}{\partial z}}{\left|\frac{\partial \vec{r}}{\partial z}\right|} = \frac{\vec{k}}{1} = \vec{k}$.

(v) Orthogonality of the System .

$\vec{e}_\rho \cdot \vec{e}_\theta = -\sin\theta\cos\theta + \sin\theta\cos\theta = 0$

$\vec{e}_\rho \cdot \vec{e}_z = 0$ and $\vec{e}_\theta \cdot \vec{e}_\rho = 0$.

Cylindrical coordinate system is an orthogonal coordinate system.

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(vi) Arc length element :

$ds = \sqrt{h_\rho^2 d\rho^2 + h_\theta^2 d^2\theta + h_z^2 d^2z}$ remember that $h_\rho = 1$ so $h_\rho^2 = 1^2$, $h_\theta = \rho$, $h_z = 1$

$ds = \sqrt{d\rho^2 + \rho^2 d^2\theta + d^2z}$

(vii) Area elements

(a) $dA_\rho = |h_\theta \vec{e}_\theta d\theta \times h_z \vec{e}_z dz| = h_\theta h_z d\theta dz$

$\Rightarrow dA_\rho = \rho\, d\theta dz$.

(b) $dA_\theta = h_\rho h_z d\rho dz = d\rho dz$.

(viii) Volume elements

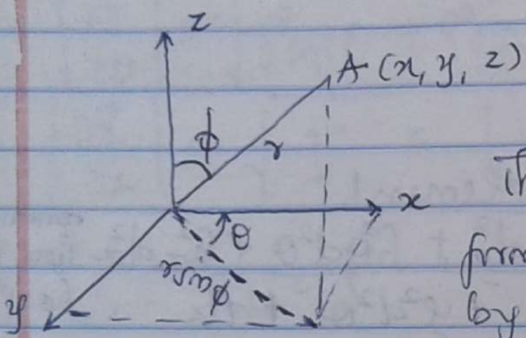
$dv^* = h_\rho h_\theta h_z\, d\rho d\theta dz$

$= \rho\, d\rho d\theta dz$

(ix) Jacobian of transformation from (x, y, z) to (ρ, θ, z) is

$$J\left(\frac{x, y, z}{\rho, \theta, z}\right) = h_\rho h_\theta h_z \left[\vec{e_\rho} \cdot (\vec{e_\theta} \times \vec{e_z})\right] J = \rho$$

B. Spherical Coordinate System.



The spherical system formed here will be defined by 3 coordinates r, ϕ and θ.

$x = r\cos\theta\sin\phi$

$y = r\sin\theta\sin\phi$

$z = r\cos\phi$

with $r \geq 0$; $0 \leq \theta < 2\pi$;

$0 \leq \phi \leq \pi$

The above gives the transformation from rectangular coordinate system (x, y, z) to spherical coordinate system (r, θ, ϕ)

1. Position Vector

To get the position vector of an arbitrary point in space, it is

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$

$$= r\cos\theta\sin\phi\,\vec{i} + r\sin\theta\sin\phi\,\vec{j} + r\cos\phi\,\vec{k}$$

2. Tangent vectors to (r, θ, ϕ) coordinates

(a) $\frac{\partial\vec{r}}{\partial r} = \cos\theta\sin\phi\,\vec{i} + \sin\theta\sin\phi\,\vec{j} + \cos\phi\,\vec{k}$

(b) $\frac{\partial\vec{r}}{\partial\theta} = -r\sin\theta\sin\phi\,\vec{i} + r\cos\theta\sin\phi\,\vec{j}$

(c) $\frac{\partial\vec{r}}{\partial\phi} = r\cos\theta\cos\phi\,\vec{i} + r\sin\theta\cos\phi\,\vec{j} - r\sin\phi\,\vec{k}$

3. Scale Factors.

(a) $h_r = \sqrt{\cos^2\theta\sin^2\phi + \sin^2\theta\sin^2\phi + \cos^2\phi}$

$= \sqrt{\sin^2\phi(\cos^2\theta + \sin^2\theta) + \cos^2\phi}$

$h_r = \sqrt{1} = 1$.

(b) $h_\theta = \sqrt{r^2\sin^2\theta\sin^2\phi + r^2\cos^2\theta\sin^2\phi}$

$= r\sin\phi$.

(c) $h_\phi = \sqrt{r^2\cos^2\theta\cos^2\phi + r^2\sin^2\theta\cos^2\phi + r^2\sin^2\phi}$

$= \sqrt{r^2\cos^2\phi(\cos^2\theta + \sin^2\theta) + r^2\sin^2\phi}$

$= \sqrt{r^2(\cos^2\phi + \sin^2\phi)}$

$= r$

Exercise

Following that, of the cylindrical coordinates obtain i) unit tangent vectors ii) show orthogonality of the spherical coordinate system.

(3) obtain i) Arc length element ii) Area element iii) volume element

iv. $J\left(\frac{x, y, z}{r, \theta, \phi}\right)$

Do the same for the parabolic cylindrical coordinate system defined as:

$x = \frac{1}{2}(u^2 - v^2)$ $y = uv$ and

$z = z$

with $-\infty < u < \infty$

$v \geq 0$ and $-\infty < z < \infty$

Example: Given that $\vec{a}$ defined in the rectangular cartesian system as

$$\vec{a} = z\vec{i} - 2x\vec{j} + y\vec{k}$$

Obtain the component of $\vec{a}$ in the cylindrical coordinate system where

$$\vec{a} = a_\rho \vec{e}_\rho + a_\theta \vec{e}_\theta + a_z \vec{e}_z$$

The unit vect $\vec{e}_\rho$, $\vec{e}_\theta$ and $\vec{e}_z$ in the cylindrical coordinate system are defined as

$$\vec{e}_\rho = \cos\theta\vec{i} + \sin\theta\vec{j} \quad \text{--- (1)}$$

$$\vec{e}_\theta = -\sin\theta\vec{i} + \cos\theta\vec{j} \quad \text{--- (2)}$$

$$\vec{e}_z = \vec{k} \quad \text{--- (3)}$$

Multiply (1) by $\cos\theta$ and (2) by $\sin\theta$ and then subtract

$$\cos\theta\,\vec{e}_\rho = \cos^2\theta\vec{i} + \cos\theta\sin\theta\vec{j}$$

$$- \quad \sin\theta\,\vec{e}_\theta = -\sin^2\theta\vec{i} + \sin\theta\cos\theta\vec{j}$$

$$\vec{i} = \cos\theta\vec{e}_\rho - \sin\theta\vec{e}_\theta \quad \text{--- (4)}$$

To obtain $\vec{j}$, multiply (1) by $\sin\theta$ and (2) by $\cos\theta$ and add.

$$\vec{j} = \sin\theta\vec{e}_\rho + \cos\theta\vec{e}_\theta \quad \text{--- (5)}$$

Recall:

$$x = \rho\cos\theta \quad y = \rho\sin\theta \quad z = z \quad \text{--- (6)}$$

Substitute eqn (4), (5) and (6) into

$$\vec{a} = z\vec{i} - 2x\vec{j} + y\vec{k}$$

$\vec{a} = z(\cos\theta\,\vec{e}_\rho - \sin\theta\,\vec{e}_\theta) - 2\rho\cos\theta(\sin\theta\,\vec{e}_\rho + \cos\theta\,\vec{e}_\theta)$
$+ \rho\sin\theta\,\vec{e}_z$

$\vec{a} = (z\cos\theta - 2\rho\cos\theta\sin\theta)\vec{e}_\rho + (-z\sin\theta - 2\rho\cos^2\theta)\vec{e}_\theta$
$+ \rho\sin\theta\,\vec{e}_z$.

$\Rightarrow a_\rho = z\cos\theta - 2\rho\cos\theta\sin\theta$

$a_\theta = -z\sin\theta - 2\rho\cos^2\theta$

$a_z = \rho\sin\theta$

GRADIENT OF A SCALAR FIELD IN AN ARBITRARY ORTHOGONAL COORDINATE SYSTEM.

Let ϕ be a scalar field whose gradient in the rectangular cartesian coordinate system is defined

$$\nabla\phi = \vec{i}\frac{\partial}{\partial x}\phi + \vec{j}\frac{\partial}{\partial y}\phi + \vec{z}\frac{\partial}{\partial z}\phi.$$

Suppose $\phi = \phi(u, v, w)$; where (u, v, w) describes an arbitrary orthogonal coordinate system.

clearly $\frac{\partial\phi}{\partial u} = \frac{\partial\phi}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial\phi}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial\phi}{\partial z}\frac{\partial z}{\partial u}$.

$$= \left(\vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{z}\frac{\partial\phi}{\partial z}\right)\cdot\left(\vec{i}\frac{\partial x}{\partial u} + \vec{j}\frac{\partial y}{\partial u} + \vec{k}\frac{\partial z}{\partial u}\right)$$

$$= \nabla\phi \cdot \frac{\partial\vec{r}}{\partial u} = h_u \nabla\phi\cdot\vec{e}_u.$$

$$\Rightarrow \nabla\phi\cdot\vec{e}_u = \frac{1}{h_u}\frac{\partial\phi}{\partial u}$$

Similarly

$$\nabla\phi\cdot\vec{e}_v = \frac{1}{h_v}\frac{\partial\phi}{\partial v}$$

$$\nabla\phi\cdot\vec{e}_w = \frac{1}{h_w}\frac{\partial\phi}{\partial w}.$$

Note that $\nabla\phi$ can be written in terms of the curvilinear system (orthogonal) as

$$\nabla\phi = a_u\vec{e}_u + a_v\vec{e}_v + a_w\vec{e}_w$$

clearly, $\nabla\phi\cdot\vec{e}_u = a_u\vec{e}_u\cdot\vec{e}_u + a_v\vec{e}_v\cdot\vec{e}_u + a_w\vec{e}_w\cdot\vec{e}_u$

$\nabla\phi\cdot\vec{e}_u = a_u = \frac{1}{h_u}\,\partial\phi/\partial u$.

$\nabla\phi\cdot\vec{e}_v = a_v = \frac{1}{h_v}\,\partial\phi/\partial v$

$\nabla\phi\cdot\vec{e}_w = a_w = \frac{1}{h_w}\,\partial\phi/\partial w$.

Gradient of ϕ in the orthogonal curvilinear system is

$$\nabla\phi = \frac{\vec{e}_u}{h_u}\frac{\partial\phi}{\partial u} + \frac{\vec{e}_v}{h_v}\frac{\partial\phi}{\partial v} + \frac{\vec{e}_w}{h_w}\frac{\partial\phi}{\partial w}.$$

clearly nabla operator is defined by

$$\nabla = \frac{\vec{e}_u}{h_u}\frac{\partial}{\partial u} + \frac{\vec{e}_v}{h_v}\frac{\partial}{\partial v} + \frac{\vec{e}_w}{h_w}\frac{\partial}{\partial w}$$

Example: Given the scalar field $\phi = \rho z \sin\theta$ defined in the cylindrical coordinate system (ρ, θ, z) obtain the gradient of ϕ i.e. $\nabla\phi$ in the system.

$$\nabla\phi = \frac{\vec{e}_\rho}{h_\rho}\frac{\partial\phi}{\partial\rho} + \frac{\vec{e}_\theta}{h_\theta}\frac{\partial\phi}{\partial\theta} + \frac{\vec{e}_z}{h_z}\frac{\partial\phi}{\partial z}.$$

$$= \vec{e}_\rho\frac{\partial\phi}{\partial\rho} + \frac{1}{\rho}\frac{\partial\phi}{\partial\theta}\vec{e}_\theta + \vec{e}_z\frac{\partial\phi}{\partial z}$$ partial derivatives $\partial\phi/\partial\rho$, $\partial\phi/\partial\theta$ & $\partial\phi/\partial z$

$$= z\sin\theta\,\vec{e}_\rho + z\cos\theta\,\vec{e}_\theta + \rho\sin\theta\,\vec{e}_z$$

11/09/19. DIVERGENCE OF A VECTOR FIELD AND LAPLACIAN OF A FIELD IN ~~AN~~ ORTHOGONAL SYSTEM

Recall the following identities

1. $\nabla\cdot(\nabla\phi\times\nabla\psi) = 0$
2. Under preservation of orientation
   $\vec{e}_u = \vec{e}_v\times\vec{e}_w$; $\vec{e}_v = \vec{e}_w\times\vec{e}_u$
   $\vec{e}_w = \vec{e}_u\times\vec{e}_v$
3. $\nabla\cdot(\phi\vec{V}) = \phi\nabla\cdot\vec{V} + \vec{V}\cdot\nabla\phi$
4. $\vec{e}_u\cdot\nabla\phi = \frac{1}{h_u}\frac{\partial\phi}{\partial u}$; $\vec{e}_v\cdot\nabla\phi = \frac{1}{h_v}\frac{\partial\phi}{\partial v}$
   $\vec{e}_w\cdot\nabla\phi = \frac{1}{h_w}\frac{\partial\phi}{\partial w}$

Divergence of a vector field.

Recall that in the orthogonal system (u, v, w) a vector $\vec{a}$ can be written as

$$\vec{a} = a_u\vec{e}_u + a_v\vec{e}_v + a_w\vec{e}_w$$

Divergence of $\vec{a}$ is

$$\nabla\cdot\vec{a} = \nabla\cdot(a_u\vec{e}_u) + \nabla\cdot(a_v\vec{e}_v) + \nabla\cdot(a_w\vec{e}_w)$$

Note and recall that

$$\nabla\phi = \frac{\vec{e}_u}{h_u}\frac{\partial\phi}{\partial u} + \frac{\vec{e}_v}{h_v}\frac{\partial\phi}{\partial v} + \frac{\vec{e}_w}{h_w}\frac{\partial\phi}{\partial w}.$$

replacing ϕ to u;

$$\nabla u = \frac{\vec{e}_u}{h_u}.$$

Similarly;

$$\nabla v = \frac{\vec{e}_v}{h_v} \quad \text{and} \quad \nabla w = \frac{\vec{e}_w}{h_w}$$

Clearly;

$$\vec{e}_u = \vec{e}_v\times\vec{e}_w = h_v h_w \nabla v\times\nabla w$$

$$\nabla\cdot(a_u\vec{e}_u) = \nabla\cdot(h_v h_w a_u \nabla v\times\nabla w)$$

from (3) identity; $\phi = h_v h_w a_u$ & $\vec{V} = \nabla v\times\nabla w$

$$= h_v h_w a_u \nabla\cdot(\nabla v\times\nabla w)\cdot(\nabla(h_v h_w a_u))$$

$$\nabla\cdot(a_u\vec{e}_u) = \vec{e}_v\times\vec{e}_w\cdot\nabla(h_v h_w a_u)$$

$$\nabla\cdot(a_u\vec{e}_u) = \frac{1}{h_u h_v h_w}\frac{\partial}{\partial u}(h_v h_w a_u)$$

Similarly;

$$\nabla\cdot(a_v\vec{e}_v) = \frac{1}{h_u h_v h_w}\frac{\partial}{\partial v}\left(h_u h_w a_v\right)$$

$$\nabla\cdot(a_w\vec{e}_w) = \frac{1}{h_u h_v h_w}\frac{\partial}{\partial w}\left(h_u h_v a_w\right)$$

Therefore;

$$\nabla\cdot\vec{a} = \frac{1}{h_u h_v h_w}\left[\frac{\partial}{\partial u}(h_v h_w a_u) + \frac{\partial}{\partial v}(h_u h_w a_v) + \frac{\partial}{\partial w}(h_u h_v a_w)\right]$$

For example; if $\vec{a} = a_\rho\vec{e}_\rho + a_\theta\vec{e}_\theta + a_z\vec{e}_z$ is a vector defined in a cylindrical coordinate system (ρ, θ, z) then the divergence of $\vec{a}$ is

$$\nabla\cdot\vec{a} = \frac{1}{\rho}\left[\frac{\partial(\rho a_\rho)}{\partial\rho} + \frac{\partial}{\partial\theta}(a_\theta) + \frac{\partial}{\partial z}(\rho a_z)\right]$$

Laplacian Of a Scalar field.

Given a scalar field defined in an arbitrary orthogonal coordinate system, we recall that the laplacian of ϕ is defined

$$\nabla^2\phi = \nabla\cdot(\nabla\phi) =$$

$$\nabla\cdot\left(\frac{\vec{e}_u}{h_u}\frac{\partial\phi}{\partial u} + \frac{\vec{e}_v}{h_v}\frac{\partial\phi}{\partial v} + \frac{\vec{e}_w}{h_w}\frac{\partial\phi}{\partial w}\right)$$

clearly.

$$\nabla^2\phi = \frac{1}{h_u h_v h_w}\left[\frac{\partial}{\partial u}\left(\frac{h_v h_w}{h_u}\frac{\partial\phi}{\partial u}\right) + \frac{\partial}{\partial v}\left(\frac{h_u h_w}{h_v}\frac{\partial\phi}{\partial v}\right) + \frac{\partial}{\partial w}\left(\frac{h_u h_v}{h_w}\frac{\partial\phi}{\partial w}\right)\right]$$

CURL OF A VECTOR IN AN ARBITRARY ORTHOGONAL SYSTEM

The curl of the vector $\vec{a} = a_u\vec{e}_u + a_v\vec{e}_v + a_w\vec{e}_w$ is defined as

$$\nabla\times\vec{a} = \frac{1}{h_u h_v h_w}\begin{vmatrix} h_u\vec{e}_u & h_v\vec{e}_v & h_w\vec{e}_w \\ \frac{\partial}{\partial u} & \frac{\partial}{\partial v} & \frac{\partial}{\partial w} \\ h_u a_u & h_v a_v & h_w a_w \end{vmatrix}$$

Example: Given that

$$\vec{a} = z\sin\theta\,\vec{e}_\rho + z\cos\theta\,\vec{e}_\theta + \rho\sin\theta\,\vec{e}_z$$

is defined in the cylindrical system (ρ, θ, z); obtain (i) $\nabla\cdot\vec{a}$ (ii) $\nabla\times\vec{a}$

solution.

1. $$\nabla\cdot\vec{a} = \frac{1}{\rho}\left[\frac{\partial}{\partial\rho}(\rho a_\rho) + \frac{\partial}{\partial\theta}(a_\theta) + \frac{\partial}{\partial z}(\rho a_z)\right]$$

where $a_\rho = z\sin\theta$, $a_\theta = z\cos\theta$, $a_z = \rho\sin\theta$

$$\nabla\cdot\vec{a} = \frac{1}{\rho}\left[\frac{\partial}{\partial\rho}(\rho z\sin\theta) + \frac{\partial}{\partial\theta}(z\cos\theta) + \frac{\partial}{\partial z}(\rho^2\sin\theta)\right]$$

$$= \frac{1}{\rho}\left[z\sin\theta - z\sin\theta\right] = 0.$$

This vector is solenoidal.

ii. $$\nabla \times \vec{a} = \frac{1}{h_\rho h_\theta h_z} \begin{vmatrix} h_\rho \vec{e}_\rho & h_\theta \vec{e}_\theta & h_z \vec{e}_z \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ h_\rho z \sin\theta & h_\theta z \cos\theta & h_z \rho \sin\theta \end{vmatrix}$$

12/09/19

COMPLEX VARIABLES.

The complex number system

There is no real number 'x' that satisfies the polynomial equation $x^2+1=0$. To permit solutions of this and similar equations, the set of complex numbers is introduced.

A complex number is a number of the form $a+bi$ where a, b belongs to $\mathbb{R}$ and i called the imaginary unit has a property that $i^2 = -1$.

If $z = a+bi$, then, a is called the real part of z and b is called the imaginary part of z and these are denoted by

$Re(z)$ and $Im(z)$ respectively.

i.e. $Re(z) = a$ and $Im(z) = b$.

The symbol z, which can be used to represent any complex number is called a complex variable.

Thus, for the earlier equation $x^2+1=0$

$\Rightarrow x^2 = -1$

$\Rightarrow x = \pm\sqrt{-1} = \pm i$

$\Rightarrow x = \pm i$

Two complex numbers $z_1 = a+bi$; $z_2 = c+di$ are equal if and only if $a=c$ and $b=d$

$Re(z_1) = Re(z_2)$ and $Im(z_1)$ and $Im(z_2)$

The set of real numbers $\mathbb{R}$ is a subset of the set of complex number $\mathbb{C}$ i.e. $\mathbb{R} \subseteq \mathbb{C}$. with $b=0$

Accordingly, the complex numbers $0+0i$ and $-4+0i$ represent the real numbers 0 and -4 respectively.

If $a=0$, the complex number $0+bi = bi$ and is called a pure imaginary number

The complex conjugate or the conjugate of a complex number $z = a+bi$ is defined as $a-bi$.

The complex conjugate or the conjugate of a complex number z is often denoted as $\bar{z}$. The complex conjugate of an real number is that real number. i.e. if $z = x \in \mathbb{R}$, then $\bar{z} = \overline{x+0i} = x-0i = x$

FUNDAMENTAL OPERATIONS WITH COMPLEX NUMBERS (ALGEBRA OF COMPLEX NUMBERS)

- To perform the operation with complex numbers, we proceed in a similar way as the algebra of real numbers, replacing

i^2 by -1 where necessary

1. Addition: Let $z_1, z_2 \in \mathbb{C}$. Then $z_1 = a+bi$ and $z = c+di$, $a, b, c, d \in \mathbb{R}$

$$z_1 + z_2 = (a+bi)+(c+di)$$
$$= (a+c)+(b+d)i$$

$Re(z_1+z_2) = Re(z_1) + Re(z_2)$

$Im(z_1+z_2) = Im(z_1) + Im(z_2)$

2. Subtraction: Let $z_1, z_2 \in \mathbb{C}$. Then $z = a+bi$ and $z = c+di$

$$z_1 - z_2 = (a+bi) - (c+di)$$
$$= (a-c) + (b-d)i$$

$Re(z_1 - z_2) = Re(z_1) - Re(z_2)$

$Im(z_1 - z_2) = Im(z_1) - Im(z_2)$

3. Multiplication: Let $z_1, z_2 \in \mathbb{C}$ then $z = a+bi$ and $z = c+di$

$$z_1 z_2 = (a+bi)(c+di) = ac + adi + bic + bdi^2$$
$$= (ac-bd) + (ad+bc)i$$

$Re(z_1 z_2) = ac - bd$ and $Im(z_1 z_2) = (ad+bc)i$

4. Division: Let $z_1, z_2 \in \mathbb{C}$. Then $z = a+bi$ and $z = c+di$

if c and d are not equal to 0, then z_1 and z_2

$$\frac{z_1}{z_2} = \frac{(a+bi)}{(c+di)}\frac{(c-di)}{(c-di)} = \frac{ac - adi + cbi - bdi^2}{c^2 - cdi + cdi - d^2}$$

$$= \frac{(ac+bd) + (bc-ad)i}{(c^2+d^2)} = \frac{ac+bd}{c^2+d^2} + \frac{(bc-ad)i}{c^2+d^2}$$

$Re\left(\frac{z_1}{z_2}\right) = \frac{ac+bd}{c^2+d^2}$ and $Im\left(\frac{z_1}{z_2}\right) = \frac{(bc-ad)}{c^2+d^2}$

ABSOLUTE VALUE

The absolute value or modulus of a complex number $z = a+bi$ is defined as

$$|z| = |a+bi| = \sqrt{a^2+b^2}$$

i.e. $|z| = \sqrt{Re(z)^2 + (Im(z))^2}$

Example 1.1

If $z = 10 - 9i$, then, the modulus of $z =$

$$|z| = \sqrt{10^2 + (-9)^2} = \sqrt{100+81} = \sqrt{181}$$

If z_1, z_2, z_3 up to z_n belong to a set of complex numbers $\mathbb{C}$, the following properties hold,

1. The modulus of product of complex numbers is the product of the moduli of the complex numbers.

$|z_1 z_2| = |z_1||z_2|$ or $\left|\prod_{i=1}^{n} z_i\right| = \prod_{i=1}^{n} |z_i|$; $|z_1 z_2 \ldots z_n| = |z_1||z_2|\ldots|z_n|$

2. $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$; $z_2 \neq 0$

3. The modulus of $|z_1+z_2| \leq |z_1| + |z_2|$ or

$\left|\sum_{i=1}^{n} z_i\right| \leq \sum_{i=1}^{n} |z_i|$; $|z_1+z_2+\ldots+z_n| \leq |z_1|+|z_2|+\ldots+|z_n|$

4. $|z_1 \pm z_2| \geq |z_1 - z_2|$

1. $z_1 z_2 = (ac-bd) + (ad+bc)i$

$|z_1 z_2| = \sqrt{(ac-bd)^2 + (ad+bc)^2}$

$|z_1| = \sqrt{a^2+b^2}$ and $|z_2| = \sqrt{c^2+d^2}$

$|z_1||z_2| = \sqrt{a^2+b^2}\sqrt{c^2+d^2}$

$= \sqrt{(a^2+b^2)(c^2+d^2)}$

$= \sqrt{a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2}$

$= \sqrt{(ac)^2 + (ad)^2 + (bc)^2 + (bd)^2}$

$= \sqrt{(ac-bd)^2 + 2acbd + (ad+bc)^2 - 2acbd}$

$(ac-bd)^2 = (ac)^2 + (bd)^2 - 2acbd$

$(ad+bc)^2 = (ad)^2 + (bc)^2 - 2abcd$

Another axiomatic formulation of complex number system.

From a strict logical point of view, we can define a complex numbers as another pair (a, b) of the real numbers a & b subject to certain operational definitions which turn out to be equivalent to the ones above. i.e. $z = a + ib$ can be regarded as $z = (a, b)$

Equality of complex numbers

If $z_1 = a + bi$ and $z_2 = c + di$, then $z_1 = z_2 \Rightarrow$

$(a, b) = (c, d)$. $\Rightarrow$ $a = c$ and $b = d$

Sum;

If $z_1 + z_2 = (a, b) + (c, d) = (a+c, b+d)$

Product;

$z_1 z_2 = (a, b)(c, d) = (ac-bd, ad+bc)$

$m(a, b) = (ma, mb)$

Theorem 1.1; Suppose $z_1, z_2, z_3 \in \mathbb{C}$, then;

1. $z_1 + z_2$, $z_1 z_2 \in \mathbb{C}$ — closure law
2. $z_1 + z_2 = z_2 + z_1$ — Commutative law of addition
3. $z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$ — Associative law of addition
4. $z_1 z_2 = z_2 z_1$ — Commutative law of multiplication
5. $z_1(z_2 z_3) = (z_1 z_2)(z_3)$ — Associative law of multiplication
6. $z_1(z_2 + z_3) = (z_1 z_2 + z_1 z_3)$ — Distributive law of addition.
7. $z_1 + 0 = 0 + z_1 = z_1$ and — Identity element

$1 \cdot z_1 = z_1 \cdot 1 = z_1$

0 is called additive identity while z is multiplicative identity

For any complex no. z_1, there exists a unique $z \in \mathbb{C}$ such that $z + z_1 = 0$

For any $z_1 \in \mathbb{C}$, there exists a unique $z \in \mathbb{C}$ such that $z_1 z = z z_1 = 1$

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Examples:

(a) $(3+2i)+(-7+i) = 3-7+2i+i = -4+3i$

(b) $(-7-i)+(3+2i) = -7+3-i+2i = -4+i$

(c) $(8-6i)-(2i-7) = 8+7-6i-2i = 15-8i$

(d) $(5+3i)+[(-1+2i)+(7-5i)] = (5+3i)+[-1+2i+7-5i]$

$(5+3i)+(6-3i) = 11$

(e) $(2-3i)(4+2i) = 2(4+2i) - 3i(4+2i)$

$= 8+4i-12i-6i^2 = 8+4i-12i+6 = 14-8i$

(f) $(2-i)[(-3+2i)(5-4i)] = (2-i)[-15+12i+10i-8i^2]$

$= (2-i)(-7+22i) = -14+44i+7i-22i^2 = 8+51i$

(g) $(-1+2i)[(7-5i)+(-3+4i)] = (-1+2i)(4-i)$

$= -4+i+8i-2i^2 = -i+9i$

(h) $\frac{3-2i}{-1+i} = \frac{3-2i}{-1+i}\frac{(-1-i)}{(-1-i)} = \frac{-3-3i+2i+2i^2}{1-i^2} = \frac{-5-i}{2}$

$= -\frac{5}{2} - \frac{1}{2}i$

(i) $\frac{5+5i}{3-4i} + \frac{20}{4+3i} = \frac{5+5i}{3-4i}\frac{(3+4i)}{(3+4i)} + \frac{20}{4+3i}\frac{(4-3i)}{(4-3i)}$

$= \frac{15+20i+15i+20i^2}{9-16i^2} + \frac{80-60i}{16-9i^2} = \frac{-5+35i}{25} + \frac{80-60i}{25}$

$= 3-i$

(j) $\frac{3i^{30}-i^{19}}{2i-1} = \frac{3(i^2)^{15}-(i^2)^9 i}{2i-1} = \frac{3(-1)^{15}-(-1)^9 i}{-1+2i}$

$\frac{-3+i}{-1+2i}\frac{(-1-2i)}{(-1-2i)} = \frac{3+6i-i-2i^2}{1-4i^2} = \frac{5+5i}{5} = 1+i$

2. Suppose $z_1 = 2+i$, $z_2 = 3-2i$ and $z_3 = -1/2 + \sqrt{3}/2i$ evaluate the following

(a) $|3z_1 - 4z_2| = |3(2+i) - 4(3-2i)|$

$= |6+3i-12+8i| = |-6+11i| = \sqrt{(-6)^2+11^2} = \sqrt{157}$

(b) $z_1^3 - 3z_1^2 + 4z_1 - 8 = (2+i)^3 - 3(2+i)^2 + 4(2+i) - 8$

$= [2^3+3\cdot2^2 i+3\cdot2i^2+i^3] - 3(4+4i+i^2) + 8+4i-8$

$= 8+12i-6-i-12-12i+3+8+4i-8 = 7+3i$

(c) $(\bar{z}_3)^4 = \left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)^4 = \left[\left(\frac{-1}{2}-\frac{\sqrt{3}i}{2}\right)^2\right]^2 = \left[\frac{1}{4}+\frac{\sqrt{3}}{2}i+\frac{3}{4}i^2\right]^2$

$= -\frac{1}{2} + \frac{\sqrt{3}}{2}i$

(d) $\left|\frac{2z_2 - z_1 - 5 - i}{2z_2 - z_2 + 3 - i}\right|^2 = \left|\frac{2(3-2i)+(2+i)-5-i}{2(2+i)-(3-2i)+3-i}\right|^2$

$= \left|\frac{3-4i}{4+3i}\right|^2 = \frac{|3-4i|^2}{|4+3i|^2} = \frac{\left(\sqrt{3^2+(-4)^2}\right)^2}{\left(\sqrt{4^2+3^2}\right)^2}$

$= 1$

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3. Find the real numbers x and y such that

$3x + 2iy - ix + 5y = 7 + 5i$

Solution

$3x + 5y + (2y - x)i = 7 + 5i$

$3x + 5y = 7$

$(2y - x)i = 5i$ By $x = -1$ $y = 2$

4. Show that the conjugate $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$

$z_1 = x_1 + y_1 i$ $\quad z_2 = x_2 + y_2 i$

GRAPHICAL REPRESENTATION OF COMPLEX NUMBERS

Since the complex number $z = x + yi$ can be considered as an ordered pair of real numbers $z = (x, y)$, we can represent such no like points in the x-y plane called the complex plane or Argand diagram. The complex no represented by P. for example could be read as either $(3,4)$ or $3 + 4i$

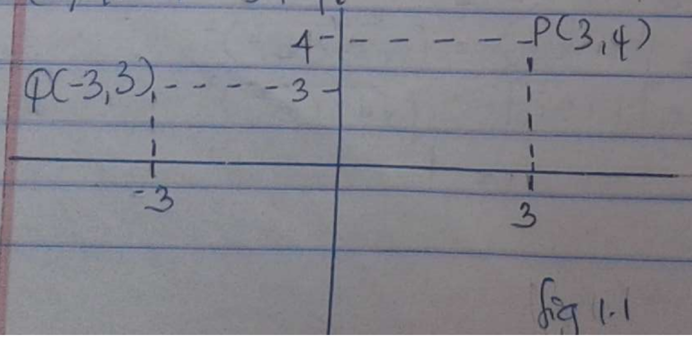


fig 1.1

To each complex no there corresponds one and only one point in the plane and conversely to each point in the plane, there corresponds one and only one complex no. As a result of this we often refer to complex no z as the point z. Sometimes, we refer to x and y axes as the real and imaginary axes respectively, and to the complex plane as z plane.

The distance between two points $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ in the complex plane is given by

$$|z_1 - z_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

POLAR FORMS OF COMPLEX NUMBERS

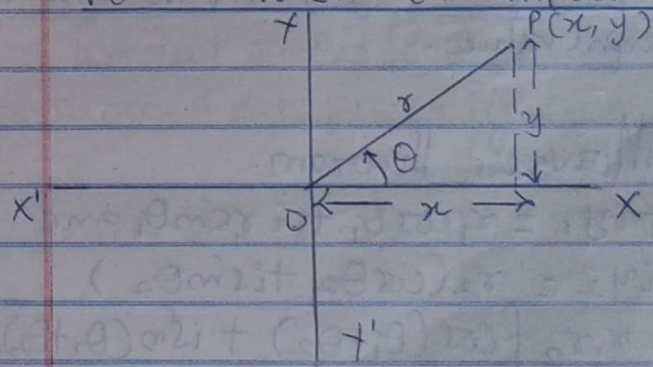


fig 1.2

Let P be a point in the complex plane corresponding to the complex no $z = x + iy$ or $z = (x, y)$. Then we can see from fig 1.2 that $x = r\cos\theta$, $y = r\sin\theta$ where;

$$r = \sqrt{x^2 + y^2} = |x + yi| = |z|$$

is called the modulus or absolute value of z and θ

called the amplitude/argument of z ($Arg(z)$), is the angle that the line OP makes with the positive x-axis (anti-clockwisely). It follows that $z = r\cos\theta + ir\sin\theta$

$$= r(\cos\theta + i\sin\theta) \quad \text{—— (1.1)}$$

This is called the polar form of the complex number. r and θ are called the polar coordinates. It is sometimes convenient to write the abbreviation $Cis\theta$ for $\cos\theta + i\sin\theta$.

For any complex no $z \neq 0$, there corresponds only one value of θ in $0 \leq \theta < 2\pi$. However, any other interval of length 2π eg. $-\pi \leq \theta \leq \pi$ can be used.

• Any particular choice decided upon in advance is called the principal range, and the value of θ is called its principal value.

De Moivre's Theorem.

Let $z_1 = x_1 + iy_1 = r_1\cos\theta_1 + ir_1\sin\theta_1$ and

$z_2 = x_2 + iy_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

then $z_1 z_2 = r_1 r_2[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)]$ — (1.2)

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)] \quad \text{—— (1.3)}$$

A generalization of 1.2 leads to $z_1 z_2 \dots z_n$

$$= r_1 r_2 \dots r_n(\cos(\theta_1+\theta_2+\dots+\theta_n) + i\sin(\theta_1+\theta_2+\dots+\theta_n)) \quad \text{—— (1.4)}$$

If $z_1 = z_2 = \dots = z_n = z$ then we'll get

$$z^n = (r\cos\theta + i\sin\theta)^n = r^n(\cos n\theta + i\sin n\theta) \quad \text{—— (1.5)}$$

This last identity is called De Moivre's Theorem.

Proof

$z_1 z_2 = r_1(\cos\theta_1 + i\sin\theta_1) r_2(\cos\theta_2 + i\sin\theta_2)$

$= r_1 r_2(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)$

$= r_1 r_2[(\cos\theta_1\cos\theta_2 + i\sin\theta_2\cos\theta_1 + i\sin\theta_1\cos\theta_2 + i^2\sin\theta_1\sin\theta_2$

$= r_1 r_2[(\cos\theta_1\cos\theta_2 + i(\sin\theta_2\cos\theta_1 + \sin\theta_1\cos\theta_2) - \sin\theta_1\sin\theta_2$

$= r_1 r_2[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)$

$= r_1 r_2[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)]$

$$\frac{z_1}{z_2} = \frac{r_1[\cos\theta_1 + i\sin\theta_1][\cos\theta_2 - i\sin\theta_2]}{r_2[\cos\theta_2 + i\sin\theta_2][\cos\theta_2 - i\sin\theta_2]}$$

$$= \frac{r_1}{r_2}\left[\frac{(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2)}{\cos^2\theta_2 + \sin^2\theta_2}\right]$$

$$= \frac{r_1}{r_2}[\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)]$$

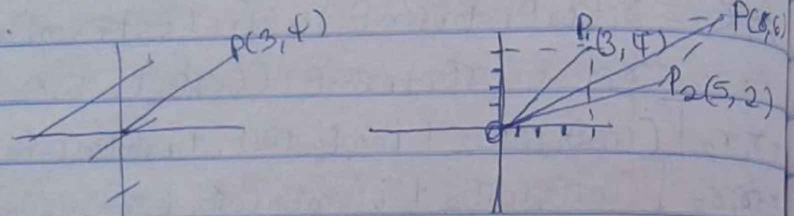
Perform the indicated operations both analytically & graphically

(a) $(3+4i) + (5+2i)$ (b) $(6-2i) - (2-5i)$

Solution

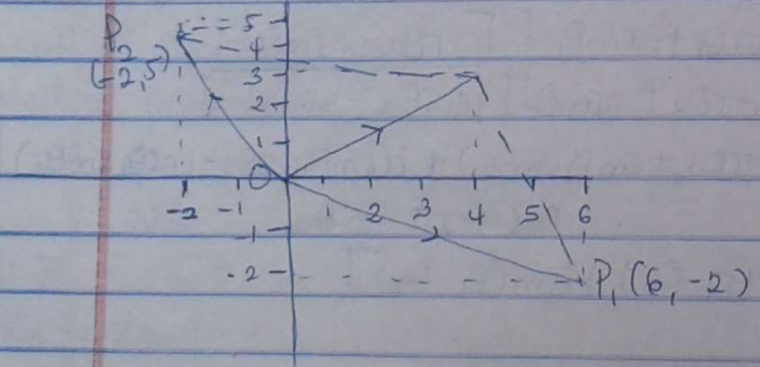
(a) Analytically $(3+4i) + (5+2i) = 8+6i$

Graphically.

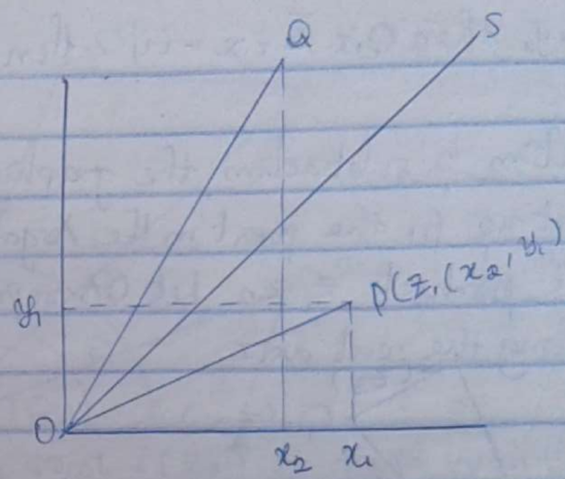


(b) $(6-2i) - (2-5i) = 4+3i$

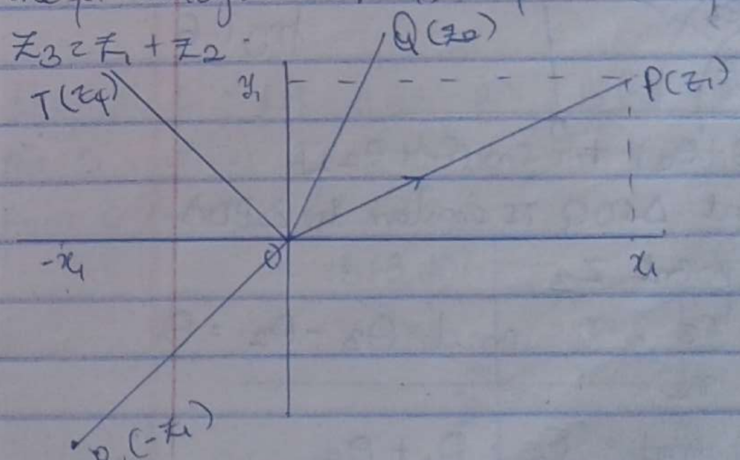
$6-2i + (-2+5i)$



The point in Argand diagram representing the (sum, difference) of 2 complex no z_1 & z_2 can be shown by simple graphical construction.



In fig 1.3, P represents z_1 and Q represents z_2. Completing the parallelogram OPSQ, the point S represents complex no $z_3 = z_1 + z_2$.



The point R represents $-z_1$ and so the point $T = z_2 + (-z_1) = z_2 - z_1$

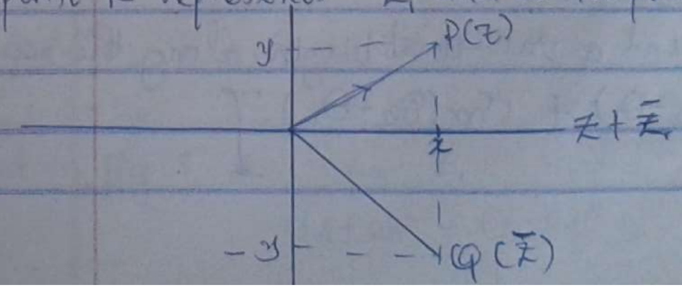


fig 1.5

If P is $z = x+iy$, then Q is $\bar{z} = x-iy$. Hence, $z+\bar{z}$ is $= 2x$.

~~$z - \bar{z} = 2x$~~

As with addition & subtraction the graphical construction can also be set up for the point in the Argand diagram representing the product $z_1 z_2$. Let OA represent one unit length along the real axis

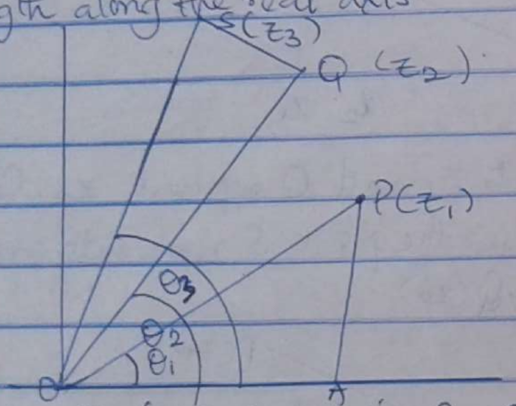


fig 1.6

$z_1 z_2 = r_1 r_2 [\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)]$

Draw S such that ΔSOQ is similar to ΔPOA

Then if S is the point z_3,

$$\frac{r_3}{r_2} = \frac{r_1}{1} \quad \text{and} \quad \theta_3 - \theta_2 = \theta_1$$

$\therefore r_3 = r_1 r_2$ and $\theta_3 = \theta_1 + \theta_2$

Therefore, $z_3 = z_1 z_2$

Let OA represent a one-unit length along the real axis

$$\frac{z_2}{z_1} = \frac{r_2}{r_1}\left[\cos(\theta_2-\theta_1) + \sin(\theta_2-\theta_1)\right]$$

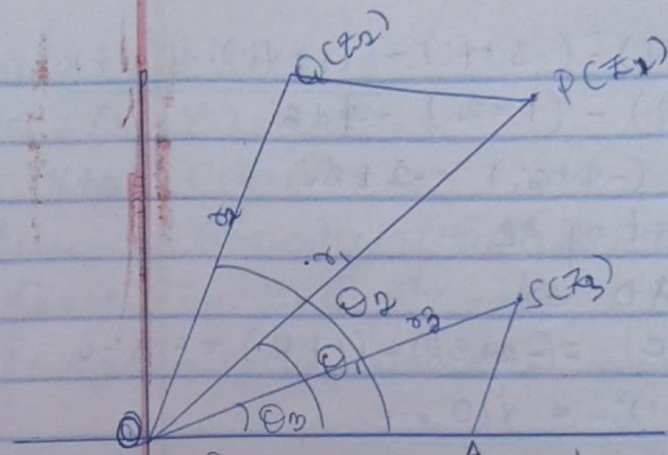


fig 1.7

In fig 1.7, the point S(z_3) is drawn such that ΔSOA is similar to ΔQOP. Consequently, $r_3/1 = r_2/r_1$ and $\theta_2 - \theta_1 = \theta_3$

therefore $z_3 = z_2/z_1$.

Example 2: Let A(1, -2), B(-3, 4) and C(2, 2) be the 3 vertices of a ΔABC. Find the length of the median from C to the side AB

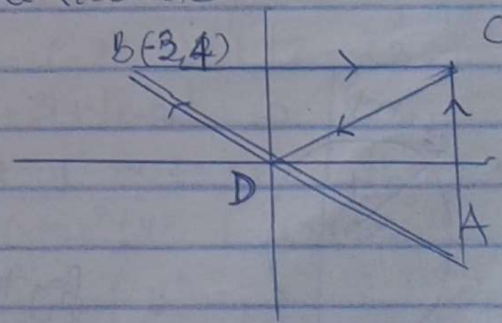


Fig 1.8

The position vectors of ABC are given by

$z_1 = 1-2i \qquad z_2 = -3+4i \qquad z_3 = 2+2i$

Based on fig 1.8, $AC = z_3 - z_1$

$= (2+2i) - (1-2i)$

$-BC = z_3 - z_2 = (2+2i) - (-3+4i) = (1+4i)$.

$AB = z_2 - z_1 = (-3+4i) - (1-2i) = -4+6i$

$AD = 1/2\, AB = 1/2\,(-4+6i) = -2+3i$

Since D is the midpoint of AB

$AC + CD = AD \Rightarrow$

$CD = AD - AC = (-2+3i) - (1+4i) = -3-i$.

$\Rightarrow |CD| = \sqrt{(-3)^2 + (-1)^2} = \sqrt{10}$.

Find an equation for

a) A circle of radius 4 with centre at $(-2, 1)$

b) An ellipse with major axis of length 10 and foci at $(-3, 0)$ and $(3, 0)$

Solution

The centre can be represented by a complex number based on the diagram below

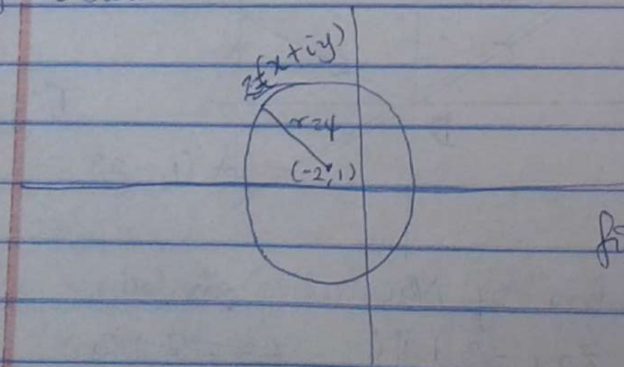


fig 1.19.

a) $|z - (-2+i)| = 4$

$|(x+iy) - (-2+i)| = 4$

$\Rightarrow |(x+2) + i(y-1)| = 4$

$= \sqrt{(x+2)^2 + (y-1)^2} = 4$

$(x+2)^2 + (y-1)^2 = 16$

Remark: Generally, if the centre of the circle is $z_0 = x_0 + iy_0$ and the radius of the circle is r, then the equation of the circle is $|z - z_0| = r$ where $z = x+iy$

b.

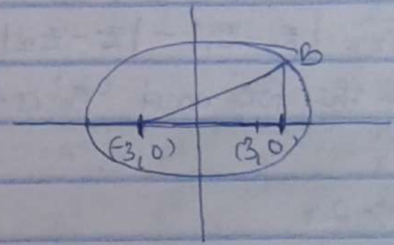


fig 1.10

The sum of the distances from any point z on the ellipse in fig 1.10 to the foci must be equal to 10, where 10 is the length of the major axis.

$|z - (-3+0i)| + |z - (3+0i)| = 10$

$|z+3| + |z-3| = 10$

$\cancel{x+iy}\ |(x+3) + yi| + |(x-3) + yi| = 10$

$\Rightarrow \sqrt{(x+3)^2 + y^2} + \sqrt{(x-3)^2 + y^2} = 10$

$= \dfrac{x^2}{25} + \dfrac{y^2}{16} = 1$ } in rectangular (x, y) plane.

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Parabola : $|z - z_0|^2 = |z - t|^2$

where z_0 is the focus and t, its directives.

Equation of an ellipse : $|z - z_1| + |z - z_2| = 2a$

where z_1 and z_2 are the foci and 'a' is the length of the semi major axis

Note that $|z_1 - z_2| < 2a$

Equation of hyperbola : $|z - z_1| - |z - z_2| = 2a$

where z_1 and z_2 are the foci and 'a' is the length of the semi major axis.

Note that $|z_1 - z_2| > 2a$

ROOTS OF COMPLEX NUMBERS.

A complex number w is called an nth root of a complex number z if $w^n = z$, and so we write

$$w = z^{1/n}$$

From De Moivre's Theorem, it can be shown if n is a +ve integer, then

$$z^{1/n} = [r(\cos\theta + i\sin\theta)]^{1/n}$$
$$= r^{1/n}\left\{\cos\left(\frac{\theta + 2k\pi}{n}\right) + i\sin\left(\frac{\theta + 2k\pi}{n}\right)\right\}$$

$k = 0, 1, 2, \ldots, n-1$ ——— 1.6.

from which it follows that there are n different values for $z^{1/n}$, i.e. n different nth roots of z, provided $z = 0$

EULER'S FORMULA

Consider the following series based on Taylor's expansion in MTH201

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots = \sum_{r=0}^{\infty} \frac{x^r}{r!}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots = \sum_{r=0}^{\infty} (-1)^{r+1} \frac{x^{2r-1}}{(2r-1)!}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots \sum_{r=0}^{\infty} (-1)^{r+1} \frac{x^{2r}}{(2r)!}$$

for $0 \le |x| < \infty$

$$e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} \ldots$$

$$= 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} \ldots$$

$$= \left[1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \ldots\right] + i\left[\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \ldots\right]$$

$$= \cos\theta + i\sin\theta$$

$$e^{i\theta} = \cos\theta + i\sin\theta \quad \text{——— 1.7}$$

Eqn 1.7 is called the Euler's formula. In general,

$$e^z = e^{x+iy} = e^x \cdot e^{iy} = e^x(\cos y + i\sin y)$$

Thus, for $z = x+iy$,

$$z = r(\cos\theta + i\sin\theta)$$

$$z = re^{i\theta}$$

Based on this new form, De Moivre's theorem can easily be verified viz

$$(e^{i\theta})^n = e^{in\theta} = \cos n\theta + i\sin n\theta$$

Also, $z_1 z_2 = r_1(\cos\theta_1 + i\sin\theta_1)\, r_2(\cos\theta_2 + i\sin\theta_2)$

$$= r_1 e^{i\theta_1} \cdot r_2 e^{i\theta_2} = r_1 r_2 e^{i(\theta_1+\theta_2)}$$

$$= r_1 r_2(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2))$$

$$z_1 z_2 = r_1 r_2 e^{i(\theta_1+\theta_2)}$$

$$\frac{z_1}{z_2} = \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)} = \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \frac{r_1}{r_2} e^{i(\theta_1-\theta_2)}$$

$$= \frac{r_1}{r_2}\left[\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)\right]$$

$$\Rightarrow z_1/z_2 = r_1/r_2\, e^{i(\theta_1-\theta_2)}$$

A transcendental function is a function that doesn't satisfy a polynomial equation, in contrast to an algebraic function which satisfies a polynomial equation. Examples include;

- Exponential function, Logarithmic function and circular function (Trigonometric function)

Polynomial Equations

In practice, we require solutions of polynomial equations having the form

$$a_0 z^n + a_1 z^{n-1} + a_2 z^{n-2} + \dots + a_{n-1} z + a_n = 0 \quad \text{(1.9)}$$

where $a_0 \neq 0, a_1, \dots, a_n$ are complex numbers and n is a +ve integer called the degree of the equation.

Such solutions are also called zeros of the polynomial on the left hand side of the polynomial 1.9 or roots of the equation. A very important theorem called the fundamental theorem of algebra states that every polynomial equation of the form (1.9) has at least one root in the complex plane. From this, we can show that it has ~~a part~~ in fact n complex roots, some (all of which may be identical.

If $z_1, z_2, \dots, z_n$ are n roots, then (1.9) can be written as

$$a(z-z_1)(z-z_2)\dots(z-z_n) = 0 \quad \text{(1.10)}$$

which is called the factored form of the polynomial equation.

The nth roots of Unity

The solution of the equation $z^n = 1$ where n is a +ve integer are called the nth roots of unity and are given by

$$z = \cos\left(\frac{2k\pi}{n}\right) + i\sin\left(\frac{2k\pi}{n}\right) \quad \text{(1.11)}$$

$$= e^{\frac{2k\pi i}{n}} \qquad k = 0, 1, 2, 3, \dots, n-1$$

If we let $\omega = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n} = e^{i\frac{2\pi}{n}}$, the n roots are $1, \omega, \omega^2, \ldots, \omega^{n-1}$

Geometrically, they represent the n vertices of a regular polygon of n sides inscribed in a circle of radius 1 with centre at the origin. This circle has the equation $|z|=1$ and is often called the unit circle.

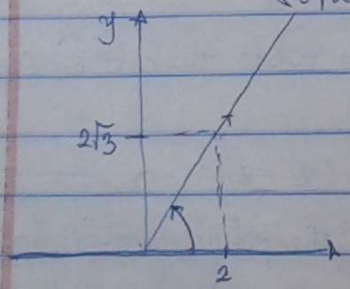
Example 3:

Express each of the following complex numbers in polar form and Euler form

(a) $2+2\sqrt{3}i$ (b) $-5+5i$ (c) $-\sqrt{6}-\sqrt{2}i$ (d) $-3i$

Solution:

(a)



$x = r\cos\theta \Rightarrow 2 = 4\cos\theta \therefore \cos^{-1}\frac{1}{2} = 60^\circ$

$y = r\sin\theta \Rightarrow 2\sqrt{3} = 4\sin\theta \therefore \sin^{-1}\frac{\sqrt{3}}{2} = 60^\circ$

$y/x = \tan\theta \Rightarrow \frac{2\sqrt{3}}{2} = \tan^{-1}\sqrt{3} = 60^\circ$

$r = \sqrt{2^2+(2\sqrt{3})^2} = \sqrt{4+12} = \sqrt{16} = 4$

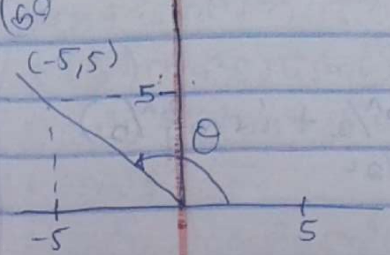
Euler form & Polar form

$2+2\sqrt{3} = 4(\cos 60^\circ + i\sin 60^\circ)$

Polar form $= 4(\cos\pi/3 + i\sin\pi/3)$

Euler form $= 4e^{i\pi/3}$

(b)



$r = |-5+5i| = \sqrt{(-5)^2+5^2}$

$= \sqrt{50} = 5\sqrt{2}$

$x = r\cos\theta \Rightarrow$

$-5 = 5\sqrt{2}\cos\theta \Rightarrow$

$\cos\theta = -5/5\sqrt{2} = -1/\sqrt{2}$

$\theta = \cos^{-1} -1/\sqrt{2} = 180-45 = 135^\circ$

In radian $= 3\pi/4$

2. $5 = 5\sqrt{2}\sin\theta$

$\sin\theta = 5/5\sqrt{2}$

$\theta = \sin^{-1} 1/\sqrt{2} = 45^\circ$

$180-45^\circ = 135^\circ$.

3. $y/x = \tan\theta$

$\tan\theta = 5/-5 = -1$

$\theta = \tan^{-1} 5/-5$

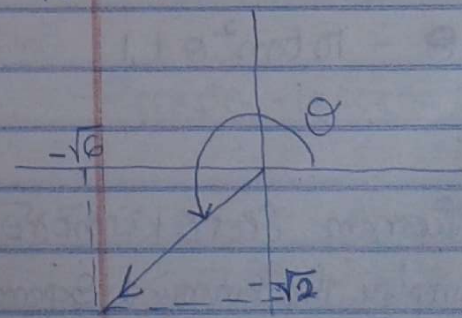
$\theta = -4$

$-5+5i = 5\sqrt{2}(\cos 135^\circ + i\sin 135^\circ)$

$= 5\sqrt{2}(\cos 3\pi/4 + i\sin 3\pi/4)$

$= 5\sqrt{2}e^{3\pi/4 i}$

(c) $-\sqrt{6}-\sqrt{2}i$



$r = |-\sqrt{6}-\sqrt{2}i| = \sqrt{6+2}$

$= \sqrt{8} = 2\sqrt{2}$

$x = r\cos\theta$: $\cos\theta = x/r = \frac{-\sqrt{6}}{2\sqrt{2}}$

$\theta = \cos^{-1}\frac{\sqrt{6}}{2\sqrt{2}} \Rightarrow 180+30 = 210^\circ$

$-\sqrt{6}-\sqrt{2}i \Rightarrow 2\sqrt{2}(\cos 7\pi/6 + i\sin 7\pi/6)$

$\Rightarrow 2\sqrt{2}e^{i7\pi/6}$

If we let $\omega = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n} = e^{i\frac{2\pi}{n}}$, the n roots are $1, \omega, \omega^2, \ldots, \omega^{n-1}$

Geometrically, they represent the n vertices of a regular polygon of n sides inscribed in a circle of radius 1 with centre at the origin. This circle has the equation $|z|=1$ and is often called the unit circle.

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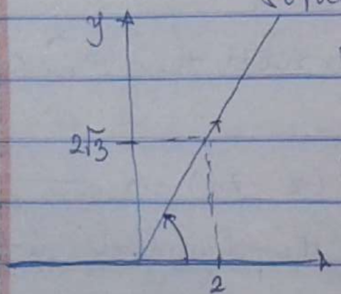
(a)



$x = r\cos\theta \Rightarrow 2 = 4\cos\theta \therefore \cos^{-1}\frac{1}{2} = 60°$

$y = r\sin\theta \Rightarrow 2\sqrt{3} = 4\sin\theta \therefore \sin^{-1}\frac{\sqrt{3}}{2} = 60°$

$y/x = \tan\theta \Rightarrow \frac{2\sqrt{3}}{2} = \tan^{-1}\sqrt{3} = 60°$

$r = \sqrt{2^2+(2\sqrt{3})^2} = \sqrt{4+12} = \sqrt{16} = 4$

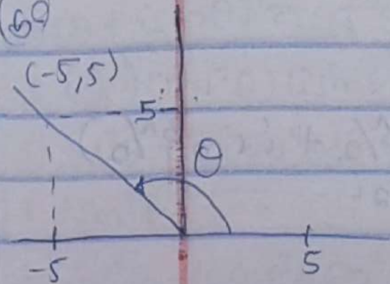
Euler form & Polar form

$2+2\sqrt{3} = 4(\cos 60° + i\sin 60°)$

Polar form $= 4(\cos\pi/3 + i\sin\pi/3)$

Euler form $= 4e^{i\pi/3}$

(b)



$r = |-5+5i| = \sqrt{(-5)^2+5^2}$

$= \sqrt{50} = 5\sqrt{2}$

$x = r\cos\theta \Rightarrow$

$-5 = 5\sqrt{2}\cos\theta \Rightarrow$

$\cos\theta = -5/5\sqrt{2} = -1/\sqrt{2}$

$\theta = \cos^{-1} 1/\sqrt{2} = 180 - 45 = 135°$

In radian $= 3\pi/4$.

2. $5 = 5\sqrt{2}\sin\theta$

$\sin\theta = 5/5\sqrt{2}$

$\theta = \sin^{-1} 1/\sqrt{2} = 45°$

$180 - 45° = 135°$.

3. $y/x = \tan\theta$.

$\tan\theta = 5/-5 = -1$

$\theta = \tan^{-1} 5/-5$

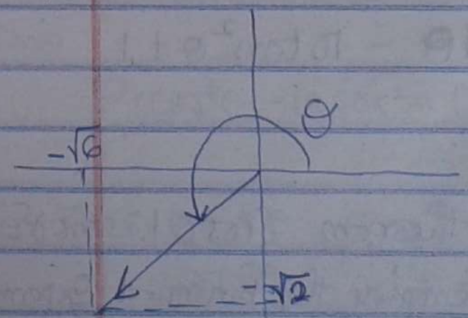
$\theta = -4$

$-5+5i = 5\sqrt{2}(\cos 135° + i\sin 135°)$

$= 5\sqrt{2}(\cos 3\pi/4 + i\sin 3\pi/4)$

$= 5\sqrt{2}e^{3\pi/4 i}$

(c) $-\sqrt{6} - \sqrt{2}i$



$r = |-\sqrt{6} - \sqrt{2}i| = \sqrt{6+2}$

$= \sqrt{8} = 2\sqrt{2}$

$x = r\cos\theta$; $\cos(\theta) = x/r = \frac{-\sqrt{6}}{2\sqrt{2}}$

$\theta = \cos^{-1}\frac{\sqrt{6}}{2\sqrt{2}} \Rightarrow 180 + 30$

$= 210° = \frac{7\pi}{6}$

$-\sqrt{6} - \sqrt{2}i \Rightarrow 2\sqrt{2}(\cos 7\pi/6 + i\sin 7\pi/6)$

$\Rightarrow 2\sqrt{2}e^{i\,7\pi/6}$

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MTH 202 NOTE

2018/19 SESSION

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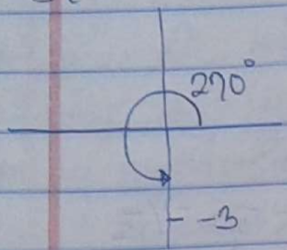
d. $-3i$



$z = -3i \Rightarrow 3\left(\cos \frac{3\pi}{2} + i \sin \frac{3\pi}{2}\right)$

$\Rightarrow 3e^{3\pi/2 i}$

Assignment

(a) $6\left(\cos 240^\circ + i \sin 240^\circ\right)$

(b) $4e^{3\pi/5 i}$

(c) $2e^{-\pi/4 i}$

5. Prove the following identities

(a) $\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$

(b) $\frac{\sin 5\theta}{\sin\theta} = 16\cos^4\theta - 12\cos^2\theta + 1$, if $\theta \neq 0, \pm\pi, \pm 2\pi$

(c) $\tan 5\theta = \frac{\tan 5\theta - 10\tan 3\theta + 5\tan\theta}{5\tan 4\theta - 10\tan^2\theta + 1}$

Solution

We shall employ De Moivre's theorem $(\cos\theta + i\sin\theta)^5 = \cos 5\theta + i\sin 5\theta$. We shall need to employ the binomial expansion $(x+y)^n = \sum_{r=0}^{n} {}^nC_r x^r y^{n-r}$ when $n = 5$

$(\cos\theta + i\sin\theta)^5 = \cos^5\theta + \binom{5}{1}\cos^4\theta(i\sin\theta) + \binom{5}{2}\cos^3\theta(i\sin\theta)^2) + \binom{5}{3}(\cos^2\theta)(i\sin\theta)^3 + \binom{5}{4}(\cos\theta)(i\sin\theta)^4 + (i\sin\theta)^5$

$= \cos^5\theta + 5i\cos^4\theta\sin\theta - 10\cos^3\sin^2\theta - 10i\cos\theta\sin^3\theta + 5\cos\theta\sin^4\theta + i\sin^5\theta$

$= \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta + i[5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta]$. Hence,

(a) $\cos 5\theta = \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$

$\Rightarrow \cos^5\theta - 10\cos^3\theta(1-\cos^2\theta) + 5\cos\theta(1-\cos^2\theta)^2$

(b) $\sin 5\theta \Rightarrow 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta$

$\frac{\sin 5\theta}{\sin\theta} \Rightarrow 5\cos^4\theta - 10\cos^2\theta\sin^2\theta + \sin^4\theta$

$\Rightarrow 5\cos^4\theta - 10\cos^2\theta(1-\cos^2\theta) + (1-\cos^2\theta)^2$

(c) $\tan 5\theta \Rightarrow \frac{\sin 5\theta}{\cos 5\theta} = \frac{5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta \,/\cos^5\theta}{\cos^5\theta - 10\cos^3\sin^2\theta + 5\cos\theta\sin^4\theta \,/\cos^5\theta}$

Divide by $\cos^5\theta$:

$$\Rightarrow \frac{5\frac{\sin\theta}{\cos\theta} - 10\frac{\sin^3\theta}{\cos^3\theta} + \frac{\sin^5\theta}{\cos^5\theta}}{1 - 10\frac{\sin^2\theta}{\cos^2\theta} + 5\frac{\sin^4\theta}{\cos^4\theta}} = \frac{\tan^5\theta - 10\tan^3\theta + 5\tan\theta}{5\tan^4\theta - 10\tan^3\theta + 1}$$

6 (a) $\cos\theta = e^{i\theta} + e^{-i\theta}$ (b) $\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$

Solution

Recall that $e^{i\theta} = (\cos\theta + i\sin\theta)$ . . . (i)

$e^{-i\theta} = \cos(-\theta) + i\sin(-\theta)$

$\Rightarrow \cos\theta - i\sin\theta$ . . . (ii)

(i) + (ii) $= e^{i\theta} + e^{-i\theta} \Rightarrow 2\cos\theta$

$\therefore \cos\theta = \frac{1}{2}\, e^{i\theta} + e^{-i\theta}$ . . . (1.12)

(b) (i) − (ii) $.\ e^{i\theta} - e^{-i\theta} = 2i\sin\theta$

$\therefore \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$ . . . (1.13).

these results i.e 1.12 & 1.13 were given by Euler in 1748, 20 yrs before Lambert introduced the hyperbolic function.

$\therefore \cosh z = \frac{1}{2}(e^{z} + e^{-z})$

$\sinh z = \frac{1}{2}.(e^{z} - e^{-z})$

Replace z by iz in cosh z and sinh z to get

$\cosh iz = \frac{1}{2}(e^{iz} + e^{-iz})$

$\sinh iz = \frac{1}{2i}(e^{iz} - e^{-iz})$

so, $\cosh iz = \cos z$ and $\sinh iz = i\sin z$

~~Since for each $i = 1, 2, 3, \ldots, n$~~

~~$x_i + y_i \in \mathbb{R}$ since $x_i, y_i \in \mathbb{R}$~~

~~$x + y = (x_1+y_1, x_2+y_2, \ldots, x_n+y_n) \in \mathbb{R}^n$~~

~~$\therefore x + y \in \mathbb{R}^n$.~~

~~2. $P_n(\mathbb{R})$ - set of all polynomials with degree less than or~~

By De Moivre's theorem

$e^{in\theta} = (e^{i\theta})^n = (\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

$\Rightarrow e^{in\theta} = \cos n\theta + i\sin n\theta$ . . . 1.16

$e^{-in\theta} = \cos n\theta - i\sin n\theta$ . . . 1.17

(1.16 + 1.17) $\Rightarrow \cos n\theta \Rightarrow \frac{1}{2}(e^{in\theta} + e^{-in\theta})$ . . . 1.18

(1.16 − 1.17) $\Rightarrow \sin n\theta = \frac{1}{2i}(e^{in\theta} - e^{in\theta})$ . . . 1.19

Prove the identities

(a) $\sin^3\theta = \frac{3}{4}\sin\theta - \frac{1}{4}\sin 3\theta$

(b) $\cos^4\theta \Rightarrow \frac{1}{8}\cos 4\theta + \frac{1}{2}\cos 2\theta + \frac{3}{8}$

Solution

(a) $\sin^3\theta = \left(\frac{e^{i\theta} - e^{-i\theta}}{2i}\right)^3 = \frac{1}{8i}\left[(e^{i\theta})^3 - 3(e^{i\theta})^2(e^{-i\theta}) + 3(e^{i\theta})(e^{-i\theta})^2 - (e^{-i\theta})^3\right]$

$= \frac{-1}{8i}\left[e^{3i\theta} - 3e^{i\theta} + 3e^{-i\theta} - 3^{-3i\theta}\right]$

$= -\frac{1}{8i}\left[e^{3i\theta} - 3e^{-3i\theta}\right] - \frac{1}{8i}\left[-3e^{i\theta} + 3e^{-i\theta}\right]$

$= -\frac{1}{4}\left(\frac{e^{3i\theta} - e^{-3i\theta}}{2i}\right)^4 + \frac{3}{4}\left(\frac{e^{i\theta} + e^{-i\theta}}{16}\right)^4$

$\Rightarrow \frac{1}{16}\left[(e^{i\theta})^4 + 4(e^{i\theta})^3(e^{-i\theta}) + 6(e^{i\theta})^2(e^{-i\theta})^2 + 4(e^{i\theta})(e^{-i\theta})^3 + (e^{-i\theta})^4\right]$

$= \frac{1}{16}\left[e^{4i\theta} + 4e^{2i\theta} + 6 + 4e^{-2i\theta} + e^{-4i\theta}\right]$

$= \frac{1}{16}\left[(e^{4i\theta} + e^{-4i\theta}) + 4(e^{2i\theta} + e^{-2i\theta}) + 6\right]$

$= \frac{1}{16}(e^{4i\theta} + e^{-4i\theta}) + \frac{1}{4}(e^{2i\theta} + e^{-2i\theta}) + \frac{3}{8}$

$\Rightarrow \frac{1}{8}\left(\frac{e^{4i\theta} + e^{-4i\theta}}{2}\right) + \frac{1}{2}\left(\frac{e^{2i\theta} + e^{-2i\theta}}{2}\right) + \frac{3}{8}$

$\Rightarrow \frac{1}{8}\cos 4\theta + \frac{1}{2}\cos 2\theta + \frac{3}{8}$

20/09/19 Assignment.

1. Prove that $e^{i\theta} = e^{i(\theta + 2k\pi)}$ $k = 0, \pm 1, \pm 2, \ldots$

2. Evaluate each of the following

a. $[3(\cos 40^\circ + i\sin 40^\circ)][4(\cos 80^\circ + i\sin 80^\circ)]$

b. $\frac{(2\,cis\,15^\circ)^7}{(4\,cis\,45^\circ)^3}$ (c) $\left(\frac{1+\sqrt{3}i}{1-\sqrt{3}i}\right)^{10}$

3. Show that

a) $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$

b) $\arg(z_1/z_2) = \arg(z_1) - \arg(z_2)$

Find all the values of z for which $z^5 + 32 = 0$.
Locate the values in the complex plane.

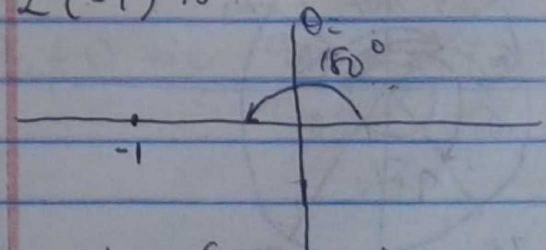
Solution

i. $z^5 + 32 = 0 \Leftrightarrow$

$z^5 = -32 \Leftrightarrow z = (-32)^{1/5}$

$= [(-1) \times 32]^{1/5} = 32^{1/5} \cdot (-1)^{1/5}$

$= 2(-1)^{1/5}$



$-1 = r(\cos\theta + i\sin\theta)$

$r = |-1| = \sqrt{(-1)^2} = 1$

$x = r\cos\theta \Rightarrow \cos\theta = \frac{x}{r} = -\frac{1}{1}$

$\theta = \cos^{-1}(-1) = 180^\circ - 0 = \pi$.

$\theta = \pi + 2k\pi$

$k = 0, \pm 1, \pm 2, \ldots$

$-1 = 1(\cos(\pi + 2k\pi) + i\sin(\pi + 2k\pi))$

$k = 0, \pm 1, \pm 2, \ldots$

$z = 2(-1)^{1/5} = 2[\cos(\pi + 2k\pi) + i\sin(\pi + 2k\pi)]^{1/5}$

$= 2\left(\cos\left(\frac{\pi + 2k\pi}{5}\right) + i\sin\left(\frac{\pi + 2k\pi}{5}\right)\right] = 2e^{i\left(\frac{\pi + 2k\pi}{5}\right)}$

$k = 0, \pm 1, \pm 2, \ldots$

if $k = 0$, $z = z_1 = 2(\cos \pi/5 + i\sin \pi/5)$.

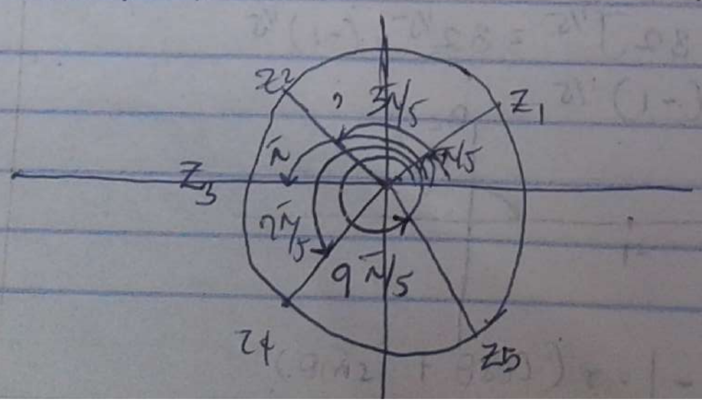
$k = 1$, $z = z_2 = 2(\cos 3\pi/5 + i\sin 3\pi/5)$

$k = 2$, $\cancel{z} = z_3 = 2(\cos 5\pi/5 + i\sin 5\pi/5) = -2$

$k = 3$, $\cancel{z_5} = z_4 = 2(\cos 7\pi/5 + i\sin 7\pi/5)$

$k = 4$, $\cancel{z} = z_4 = 2(\cos 9\pi/5 + i\sin 9\pi/5)$

ii



9. Find each of the indicated roots and find them graphically.

(a) $(-1+i)^{1/3}$ (b) $(-2\sqrt{3} - 2i)^{1/4}$ (c) $(-15-8i)^{1/2}$

Solution.

a. $-1 + i = \sqrt{2}\left\{\cos\left(\frac{3\pi}{4} + 2k\pi\right) + i\sin\left(\frac{3\pi}{4} + 2k\pi\right)\right\}$

$(-1+i)^{1/3} = 2^{1/6}\left[\cos\left(\frac{3\pi/4 + 2k\pi}{3}\right) + i\sin\left(\frac{3\pi/4 + 2k\pi}{3}\right)\right]$

Theorem 1:- Suppose the real rational number p/q (where p and q have no common factor except ± 1 i.e p/q is in the lowest terms) satisfies the polynomial equation

$a_0 z^n + a_1 z^{n-1} + \ldots + a_n = 0$

where $a_0, \ldots, a_n$ are integers. Show that p and q ~~must be~~ are factors of a_n and a_0 respectively.

Theorem 2:- The sum and products of all the roots of

$a_0 z^n + a_1 z^{n-1} + \ldots + a_n = 0$

where $a_0 \neq 0$ are $\frac{a_1}{a_0}$ and $\frac{(-1)^n a_n}{a_0}$

Theorem 3:- Suppose $p+qi$ is a root of
$a_0z^n + a_1z^{n-1} + \dots + a_n = 0$
where $a_0 \neq 0, a_1, \dots, a_n$, p and q are real.
Then $p-qi$ is also a root

Example:- Solve $6z^4 - 25z^3 + 32z^2 + 3z - 10 = 0$

Solution

We shall be guided by Theorem 1.
$a_0 = 6$ and $a_n = -10$
The integer factors of $\underset{q}{6}$ and $\underset{p}{-10}$ are respectively
$\pm1, \pm2, \pm3, \pm6$ and $\pm1, \pm2, \pm5, \pm10$.
Hence, by Theorem 1, the possible rational solutions
p/q are; $\pm1, \pm\frac{1}{2}, \pm\frac{1}{3}, \pm\frac{1}{6}, \pm2, \pm\frac{2}{3}, \pm5, \pm\frac{5}{2}$
$\pm\frac{5}{3}, \pm\frac{5}{6}, \pm10, \pm\frac{10}{3}$.
By trial, we find that $z = -\frac{1}{2}$ and $z = \frac{2}{3}$ are solutions and so; $(2z+1)$ and $3z-2$ are factors
So, $(2z+1)(3z-2) = (6z^2 - z - 2)$
is a factor of the polynomial. By long division
By long division,

$$6z^4 - 25z^3 + 32z^2 + 3z - 10 = (6z^2 - z - 2)(z^2 - 4z + 5) = 0$$

Solving $z^2 - 4z + 5$ by quadratic formula

$$z = \frac{4 \pm \sqrt{16 - 4 \cdot 5 \cdot 1}}{2} = \frac{4 \pm \sqrt{-4}}{2} = \frac{4 \pm 2i}{2} = 2 \pm i$$

Therefore, the solutions are;
$z = -\frac{1}{2}, \frac{2}{3}, 2+i, 2-i$.

Remark:- For the above polynomial $a_0 = 6$, $a_1 = -25$ and $a_n = -10$. So, $\frac{-a_1}{a_0} = \frac{-(-25)}{6} = \frac{25}{6}$

The sum of roots $= -\frac{1}{2} + \frac{2}{3} + 2+i + 2-i = -\frac{1}{2} + \frac{2}{3} + 4 = \frac{25}{6}$

$$\frac{(-1)^4(-10)}{6} = \frac{-10}{6} = -\frac{5}{3}.$$

Products of roots $= (-\frac{1}{2})(\frac{2}{3})(2+i)(2-i)$
$= (-\frac{1}{3})(4+1) = -\frac{5}{3}$

This confirms Theorem 2.

For Theorem 3, it will be observed that $2+i$ & $2-i$ are both roots of the above of the given polynomial

A number is said to be an algebraic number if it is a solution of a polynomial equation
$a_0z^n + a_1z^{n-1} + \dots + a_{n-1}z + a_n = 0$
where $a_0, a_1, \dots, a_n$ are integers

Example:- show that

(a) $\sqrt{3}+\sqrt{2}$ (b) $\sqrt[3]{4}-2i$ are algebraic

Solution.

(a)

Let $z=\sqrt{3}+\sqrt{2}$

Square both sides.

$z^2 = 3+2+2\sqrt{6}$.

$z^2-5 = 2\sqrt{6}$; Square both sides again:

$(z^2-5)^2 = 4\times 6$

$z^4-10z^2+1 = 0$.

(b) Let $z = \sqrt[3]{4}-2i$,

$z+2i = \sqrt[3]{4}$. Cube both sides.

$(z+2i)^3 = (\sqrt[3]{4})^3$

$z^3+3z^2(2i)+3z(2i)^2+(2i)^3 = 4$

$\Rightarrow z^3-12z-4 = i(8-6z^2)$; square both sides

$\Rightarrow (z^3-12z-4)^2 = (i(8-6z^2))^2$

$\Rightarrow z^6+12z^4-8z^3+48z^2+96z+80 = 0$

Recall $z = \cos\theta + i\sin\theta$ and so

$\frac{1}{z} = z^{-1} = \cos\theta - i\sin\theta$.

Adding both equations;

$$2\cos\theta = z+\frac{1}{z}$$

Subtracting; $2i\sin\theta = z - 1/z$.

Similarly; $z^n = (\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

and $\frac{1}{z^n} = (\cos\theta + i\sin\theta)^{-n} = \cos n\theta - i\sin n\theta$

$\Rightarrow z^n = \cos n\theta + i\sin n\theta$

$\frac{1}{z^n} = \cos n\theta - i\sin n\theta$

Adding: $2\cos n\theta = z^n + 1/z^n$

Subtracting: $2i\sin n\theta = z^n - 1/z^n$.

These results are useful in expanding the powers of $\sin\theta$ and $\cos\theta$ in terms of multiple angles.

eg. $2^6\cos^6\theta = (2\cos\theta)^6 = (z+1/z)^6$

$= \left(z^6+\frac{1}{z^6}\right)+6\left(z^4+\frac{1}{z^4}\right)+15\left(z^2+\frac{1}{z^2}\right)+20$

$2\cos 6\theta + 12\cos 4\theta + 30\cos 2\theta + 20$

$\Rightarrow \cos^6\theta = \frac{1}{32}\left[\cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10\right]$

eg 2

$2^5 i^5 \sin^5\theta = (2i\sin\theta)^5$

$= \left(z-\frac{1}{z}\right)^5$

$= \left(z^5-\frac{1}{z^5}\right) - 5\left(z^3-\frac{1}{z^3}\right) + 10\left(z-\frac{1}{z}\right)$

$= 2i\sin 5\theta - 5\cdot 2i\sin 3\theta + 10\cdot 2i\sin\theta$

Example:- Show that (a) $\sqrt{3}+\sqrt{2}$ (b) $\sqrt[3]{4}-2i$ are algebraic

Solution.

(a)

Let $z = \sqrt{3}+\sqrt{2}$

Square both sides.

$z^2 = 3+2+2\sqrt{6}$.

$z^2-5 = 2\sqrt{6}$; Square both sides again.

$(z^2-5)^2 = 4\times 6$

$z^4-10z^2+1 = 0$.

(b) Let $z = \sqrt[3]{4}-2i$,

$z+2i = \sqrt[3]{4}$. cube both sides.

$(z+2i)^3 = (\sqrt[3]{4})^3$

$z^3+3z^2(2i)+3z(2i)^2+(2i)^3 = 4$

$\Rightarrow z^3-12z-4 = i(8-6z^2)$; Square both sides

$\Rightarrow (z^3-12z-4)^2 = (i(8-6z^2))^2$

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Similarly; $z^n = (\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

and $\frac{1}{z^n} = (\cos\theta + i\sin\theta)^{-n} = \cos n\theta - i\sin n\theta$

$\Rightarrow z^n = \cos n\theta + i\sin n\theta$

$\frac{1}{z^n} = \cos n\theta - i\sin n\theta$

Adding: $2\cos n\theta = z^n + 1/z^n$

Subtracting: $2i\sin n\theta = z^n - 1/z^n$.

These results are useful in expanding the powers of $\sin\theta$ and $\cos\theta$ in terms of multiple angles.

eg. $2^6\cos^6\theta = (2\cos\theta)^6 = (z+1/z)^6$

$= \left(z^6+\frac{1}{z^6}\right) + 6\left(z^4+\frac{1}{z^4}\right) + 15\left(z^2+\frac{1}{z^2}\right) + 20z$

$2\cos 6\theta + 12\cos 4\theta + 30\cos 2\theta + 20$

$\Rightarrow \cos^6\theta = \frac{1}{32}\left[\cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10\right]$

eg 2

$2^5 i^5 \sin^5\theta = (2i\sin\theta)^5$

$= \left(z-\frac{1}{z}\right)^5$

$= \left(z^5-\frac{1}{z^5}\right) - 5\left(z^3-\frac{1}{z^3}\right) + 10\left(z-\frac{1}{z}\right)$

$= 2i\sin 5\theta - 5\cdot 2i\sin 3\theta + 10\cdot 2i\sin\theta$

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$$2^5\sin^5\theta = 2\sin 5\theta - 10\sin 3\theta + 20\sin\theta.$$

(2) $\sin^5\theta = \frac{1}{16}\left[\sin 5\theta - 5\sin 3\theta + 10\sin\theta\right]$

2 LINEAR VECTOR SPACE

Definition:- Let V be a nonempty set and $\mathbb{K}$ a field of scalars. V endowed with operation of vector addition and scalar multiplication $(V, +, \cdot)$ is said to be a linear vector space (vector space, linear space) over a field $\mathbb{K}$ if the following axioms are satisfied:

A_1: $x+y = y+x \in V \ \forall\ x, y \in V$ (Commutativity)

A_2: $(x+y)+z = x+(y+z)\ \forall\ x, y, z \in V$ (associativity)

A_3: There exists vector $\underline{0} \in V$, such that $x+\underline{0} = x = \underline{0}+x\ \forall x \in V$ (identity)

A_4:- For every $x \in V$, there exists $x \in V$, such that
$x+(-x) = \underline{0} = (-x)+x$

M_1: For each $\alpha \in \mathbb{K}$, $\alpha(x+y) = \alpha x + \alpha y$, $x, y \in V$

M_2: $(\alpha+\beta)x = \alpha x + \beta x\ \forall\ \alpha, \beta \in \mathbb{K}$

M_3: $\alpha(\beta x) = \alpha\beta(x)\ \forall\ \alpha, \beta \in \mathbb{K}$, $x \in V$

M_4: There exists a unit $1 \in \mathbb{K}$ such that $1\cdot x = x = x\cdot 1$, $\forall\ x \in V$

The field of scalars $\mathbb{K}$ is usually $\mathbb{R}$ (real numbers) or $\mathbb{C}$ (Complex numbers)

Example:-

(i) $\mathbb{R}^n$ with operations + defined as $x, y \in \mathbb{R}^n$
$$x+y = (x_1+y_1, x_2+y_2, \dots, x_n+y_n) \in \mathbb{R}^n$$
and scalar multiplication defined as
for all $x \in \mathbb{R}^n$, $\alpha \in \mathbb{R}$
$$\alpha x \Rightarrow (\alpha x_1, \alpha x_2, \dots, \alpha x_n) \in \mathbb{R}^n$$
$(\mathbb{R}^n, +, \cdot)$ is a vector space over $\mathbb{R}$, similarly $\mathbb{C}^n$ over $\mathbb{R}$ or $\mathbb{C}$ is a vector space with vector addition and scalar multiplication
$$\forall\ u, v \in \mathbb{C}^n$$
$$u = (u_1, u_2, \dots, u_n)$$
$$v = (v_1, v_2, \dots, v_n)$$
$u+v = (u_1+v_1, u_2+v_2, \dots, u_n+v_n)$ and
$$\alpha u = (\alpha u_1, \alpha u_2, \dots, \alpha u_n).\ \alpha \in \mathbb{C}$$
Note that $u_1 = x_1 + iy_1$ for some $x_1, y_1 \in \mathbb{R}$ and $i = \sqrt{-1}$.
Also for $i = 2, \dots, n$
$$\alpha u_j = \alpha x_j + i\alpha y_j \ ; j = 1, 2, \dots, n$$

Solution.
Since for each $i = 1, 2, 3, \dots n$, $x_i + y_i \in \mathbb{R}$. since $x_i + y_i \in$
$$x+y = (x_1+y_1, x_2+y_2, \dots, x_n+y_n) \in \mathbb{R}^n$$
$$x+y \in \mathbb{R}^n$$

25/09/19.

(2) $P_n(\mathbb{R})$ - set of all polynomials with degree less than or equal to n with real coefficients.

$V_1 = P_n(\mathbb{R})$ is a vector space over field $\mathbb{R}$

Vector Addition:

$p, q \in P_n(\mathbb{R})$

$p(t) = a_n t^n + a_{n-1} t^{n-1} + \dots + a_1 t + a_0$.

$q(t) = b_n t^n + b_{n-1} t^{n-1} + \dots + b_1 t + b_0$.

$(p+q)(t)$

$= p(t) + q(t)$

$= (a_n + b_n) t^n + \dots (a_1 + b_1) + (a_0 + b_0)$

Scalar Multiplication

$\alpha \in \mathbb{R}$, $p \in P_n(\mathbb{R})$

$\alpha p(t) = \alpha a_n t^n + \dots + \alpha a_1 t + \alpha a_0$

where $a_i, b_i \in \mathbb{R}$, $\forall\ i = 1, \dots, n$.

(3) Space of functions on $\mathbb{R}$; $\mathbb{F}(\mathbb{R}) = \{f : f : \mathbb{R} \to \mathbb{R}$ and f is continuous$\}$

Vector Addition:

Point wise addition of functions as:

$f, g \in \mathbb{F}(\mathbb{R})$ such that for each $t \in \mathbb{R}$

$(f+g)(t) = f(t) + g(t)$

$(f+g)(t) = f(t) + g(t)\ \forall t \in \mathbb{R}$

Scalar Multiplication:

for any $\alpha \in \mathbb{R}$, $f \in \mathbb{F}(\mathbb{R})$

$(\alpha f)(t) = \alpha f(t)$, $t \in \mathbb{R}$

So with the pointwise addition and scalar multiplication defined above.

$\mathbb{F}(\mathbb{R})$ is a vector space over field $\mathbb{R}$

26/09/19. Linear Combination

Definition: Let V be a vector space over a field of scalars $\mathbb{K}$. Let v be a vector in V and the vectors $v_1, v_2, \dots, v_m \in V$. The vector v is said to be a linear combination of $v_1, v_2, \dots, v_m$ if there exists scalars $\alpha_1, \alpha_2, \dots, \alpha_m$ such that

$v = \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_m v_m$

Example:- Express $v = (1, -2, 5) \in \mathbb{R}^3$ as a linear combination of the vectors $u_1 = (1,1,1)$, $u_2 = (1,2,3)$ and $u_3 = (2,-1,1)$

we have to find scalars $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ such that

$v = \alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3$

$\Rightarrow (1,-2,5) = \alpha_1 (1,1,1) + \alpha_2 (1,2,3) + \alpha_3 (2,-1,1)$

$= (\alpha_1, \alpha_1, \alpha_1) + (\alpha_2, 2\alpha_2, 3\alpha_2) + (2\alpha_3, -\alpha_3, \alpha_3)$

$= (\alpha_1 + \alpha_2 + 2\alpha_3,\ \alpha_1 + 2\alpha_2 - \alpha_3,\ \alpha_1 + 3\alpha_2 + \alpha_3)$

$\Rightarrow \alpha_1 + \alpha_2 + 2\alpha_3 = 1$

$\alpha_1 + 2\alpha_2 - \alpha_3 = -2$

$\alpha_1 + 3\alpha_2 + \alpha_3 = 5$

$\Rightarrow \alpha_1 = -6, \alpha_2 = 3, \alpha_3 = 2$

$\therefore v = -6u_1 + 3u_2 + 2u_3$

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2. Express $v = t^2 + 4t - 3$ as a linear combination of the polynomials
$u_1 = t^2 - 2t + 5$, $u_2 = 2t^2 - 3t$, $u_3 = t + 3$

Solution.

To find scalars $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ such that
$v = \alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3$.

$t^2 + 4t - 3 = \alpha_1 (t^2 - 2t + 5) + \alpha_2 (2t^2 - 3t) + \alpha_3 (t + 3)$

$\Rightarrow t^2 + 4t - 3 = (\alpha_1 + 2\alpha_2) t^2 + (-2\alpha_1 - 3\alpha_2 + \alpha_3) t + (5\alpha_1 + 3\alpha_3)$

$\Rightarrow \alpha_1 + 2\alpha_2 = 1$
$-2\alpha_1 - 3\alpha_2 + \alpha_3 = 4$
$5\alpha_1 + 3\alpha_3 = -3$

$\Rightarrow \alpha_1 = -3, \ \alpha_2 = 2, \ \alpha_3 = 4$.

$\therefore v = -3u_1 + 2u_2 + 4u_3$.

3. Let $D = \begin{bmatrix} 4 & 7 \\ 7 & 9 \end{bmatrix}$, $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$,
$B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 1 \\ 4 & 5 \end{bmatrix}$

Express D as a linear combination of A, B and C

Solution.

To find scalars α, β, γ such that $D = \alpha A + \beta B + \gamma C$

$\Rightarrow \begin{bmatrix} 4 & 7 \\ 7 & 9 \end{bmatrix} = \alpha \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + \beta \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \gamma \begin{bmatrix} 1 & 1 \\ 4 & 5 \end{bmatrix}$

$= \begin{bmatrix} \alpha + \beta + \gamma & \alpha + 2\beta + \gamma \\ \alpha + 3\beta + 4\gamma & \alpha + 4\beta + 5\gamma \end{bmatrix}$

$\alpha + \beta + \gamma = 4$
$\alpha + 2\beta + \gamma = 7$ $\Longrightarrow \alpha = 2, \ \beta = 3, \ \gamma = -1$
$\alpha + 3\beta + 4\gamma = 7$
$\alpha + 4\beta + 5\gamma = 9$

Linear Span.

Let V be a vector space over a field $\mathbb{K}$. Vectors $a_1, a_2, \ldots, a_n$ in V are said to span or generate V if every $v \in V$ is a linear combination. span $\{u_1, u_2, \ldots, u_n\}$ is called a linear span of V

Example:

(1) Let $e_1 = (1, 0, 0)$ $e_2 = (0, 1, 0)$ and $e_3 = (0, 0, 1)$
show that $\{e_1, e_2, e_3\}$ is a span of $\mathbb{R}^3$

Solution

Let $v \in \mathbb{R}^3 \Rightarrow v = \{u_1, u_2, u_3\}$ where $u_1, u_2, u_3 \in \mathbb{R}$
For $\{e_1, e_2, e_3\}$ to be a span of $\mathbb{R}^3$

$v = \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3$

$\Rightarrow (u_1, u_2, u_3) = \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3$.

$= \alpha_1 (1,0,0) + \alpha_2 (0,1,0) + \alpha_3 (0,0,1)$

$= \alpha_1 \alpha_2 \alpha_3$

$\alpha_1 = u_1, \quad \alpha_2 = u_2, \quad \alpha_3 = u_3$.

So. $V = u_1 e_1 + u_2 e_2 + u_3 e_3$; since $\mathbb{R}^3$ was arbitrary chosen, we conclude that;

$\Rightarrow (e_1, e_2, e_3)$ is a span of $\mathbb{R}^3$

2. Show that vectors $u_1 = \{1, 1, 1\}$ $u_2 = \{1, 2, 3\}$ $u_3 = \{1, 5, 8\}$.

$V = (v_1, v_2, v_3)$

$V = \alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3$

$(v_1, v_2, v_3) = \alpha_1 (1,1,1) + \alpha_2 (1,2,3) + \alpha_3 (1,5,8)$

$= (\alpha_1, \alpha_1, \alpha_1) + (\alpha_2, 2\alpha_2, 3\alpha_2) + (\alpha_3, 5\alpha_3, 8\alpha_3)$

$V_1 = \alpha_1 + \alpha_2 + \alpha_3$. — (1)

$V_2 = \alpha_1 + 2\alpha_2 + 5\alpha_3$ — (2)

$V_3 = \alpha_1 + 3\alpha_2 + 8\alpha_3$ — (3)

(2) − (1)

$V_2 - V_1 = \alpha_2 + 4\alpha_3$ — (4)

(3) − (2)

$V_3 - V_2 = \alpha_2 + 3\alpha_3$ — (5)

(4) − (5)

$V_2 - V_1 - (V_3 - V_2) \Rightarrow 2V_2 - V_1 - V_3$.

$2V_2 - V_1 - V_3 = \alpha_3$

$\alpha_2 = V_3 - V_2 - 3(2V_2 - V_1 - V_3)$

$\alpha_2 = 4V_3 - 7V_2 + 3V_1$

$\alpha_1 = V_1 - \alpha_2 - \alpha_3$

$= V_1 - 2V_2 + V_1 + V_3 - 4V_3 + 7V_2 - 3V_1$

$\alpha = -V_1 + 5V_2 - 3V_3$

Since $\exists\ \alpha_1, \alpha_2, \alpha_3$.

2/10/2019

(2) Show that W is not a subspace of $V = \mathbb{R}^3$ over $\mathbb{R}$ if

$W = \{(a,b,c) : a^2+b^2+c^2 \leq 1\}$

solution:

i) $\underline{0} = (0,0,0)$ since $0^2+0^2+0^2 = 0$

Let $u = (1,0,0)$

$v = (0,0,1)$

$1^2+0^2+0^2 = 1 \leq 1$ $\therefore u \in W$

Similarly, $v \in W$ since

$v = (0,0,1)$ $\ni 0^2+0^2+1^2 = 1 \leq 1$

$u+v = (1,0,1)$ $\quad 1^2+0^2+1^2 = 2 \not\leq 1$

$\Rightarrow u+v \notin W$

$\Rightarrow W$ is not a vector subspace of V

ii) $u,v \in W$, $\alpha,\beta \in \mathbb{R}$

$u = (a_1, b_1, c_1)$, $\quad a_1^2+b_1^2+c_1^2 \leq 1$

$v = (a_2, b_2, c_2)$ $\quad a_2^2+b_2^2+c_2^2 \leq 1$

$\alpha u + \beta v = (\alpha a_1 + \beta a_2, \alpha b_1 + \beta b_2, \alpha c_1 + \beta c_2)$

but

$(\alpha a_1 + \beta a_2)^2 + (\alpha b_1 + \beta b_2)^2 + (\alpha c_1 + \beta c_2)^2 \not\leq 1$

for $\alpha, \beta \geq 1$

hence, $u,v \in W$, $\alpha,\beta \in \mathbb{R}$, $\alpha u + \beta v \notin W$

Hence, W is not a vector subspace of V.

3. Let V be a vector space of real-value functions over field $\mathbb{R}$. Show that

$W = \{f \mid f : \mathbb{R} \to \mathbb{R}, f(-x) = -f(x)\}$

is a vector subspace of V.

Solution

i $\underline{0}(x) = 0 \quad \forall x \in \mathbb{R}$

$\underline{0}(-x) = 0$

$\Rightarrow \underline{0}(x) = -0(-x) = 0$

$= -\underline{0}(x) \quad \Rightarrow \underline{0} \in W$

ii. $f_1, f_2 \in W$; $\alpha, \beta \in \mathbb{R}$

$f_1 \in W \Rightarrow f_1(-x) = -f_1(x) \quad \forall x \in \mathbb{R}$

$f_2 \in W \Rightarrow f_2(-x) = -f_2(x) \quad \forall x \in \mathbb{R}$

$(\alpha f_1 + \beta f_2)(-x)$

$= \alpha f_1(-x) + \beta f_2(-x)$

$= -\alpha f_1(x) - \beta f_2(x)$

$= -(\alpha f_1 + \beta f_2)(x)$

$\Rightarrow \alpha f_1 + \beta f_2 \in W$ for any $f_1, f_2 \in W$; $\alpha, \beta \in \mathbb{R}$

(i) and (ii) $\Rightarrow W$ is a vector subspace of V

Ex.

1. Let V be a vector space of real-valued functions over field $\mathbb{R}$. Show that

$W = \{f \mid f : \mathbb{R} \to \mathbb{R}, f$ is continuous on $\mathbb{R}\}$

is a vector subspace of V

(2) Let V be a linear vector space over field $\mathbb{K}$ and let U, W be vector subspaces of V. Show that:

(a) $U + W = \{v = u + w;\ u \in U \text{ and } w \in W\}$ is a vector subspace of V and

(b) $U \cap W = \{v : v \in U \text{ and } v \in W\}$ is a vector subspace of V

Linear Dependence

Let V be a linear vector space over a field $\mathbb{K}$.
Vectors $v_1, v_2, \dots, v_n$ are said to be <u>linearly dependent</u> if there exists scalars $\alpha_1, \alpha_2, \dots, \alpha_m$ not all equal to zero such that

$$\alpha_1 u_1 + \alpha_2 u_2 + \dots + \alpha_n u_m = 0$$

It implies that if α_m (say) $\neq 0$ then

$$V_m = \left(\frac{-\alpha_1}{\alpha_m}\right)V_1 + \left(\frac{-\alpha_2}{\alpha_m}\right)V_2 + \dots + \left(\frac{-\alpha_{m-1}}{\alpha_m}\right)V_{m-1}$$

$$= \beta_1 V_1 + \beta_2 V_2 + \dots + \beta_{m-1} V_{m-1}$$

$\Rightarrow V_m$ can be expressed as a linear combination of vectors $V_1, V_2, \dots, V_{m-1}$.

Hence, $V_1, V_2, \dots, V_m \in V$ are said to be linearly dependent if one of the vectors can be expressed as a linear combination of others.

Vectors $V_1, V_2, \dots, V_m \in V$ are said to be <u>linearly independent</u> if for any scalars

$$\alpha_1, \alpha_2, \dots, \alpha_m \in \mathbb{K},$$

$$\alpha_1 V_1 + \alpha_2 V_2 + \dots + \alpha_m V_m = 0$$

only if $\alpha_1 = \alpha_2 = \dots = \alpha_m = 0$

Examples:-

Determine whether or not u and v are linearly dependent where;

(a) $u = (1, 2, -3)$, $v = (4, 5, -6)$

(b) $u = (1, -3)$, $v = (-2, 6)$

(c) $u = \begin{bmatrix} 1 & 3 & -4 \\ 5 & 0 & 1 \end{bmatrix}$, $v = \begin{bmatrix} -4 & -12 & 6 \\ -20 & 0 & 4 \end{bmatrix}$

Solution

(a) $u = (1, 2, -3)$, $v = (4, 5, -6)$

We have to find $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha u + \beta v = \underline{0}$$

$$\Rightarrow \alpha(1, 2, -3) + \beta(4, 5, -6) = (0, 0, 0)$$

$\Rightarrow$

$\alpha + 4\beta = 0$

$2\alpha + 5\beta = 0$

$-3\alpha - 6\beta = 0$

$\Rightarrow \alpha = \beta = 0$

hence for any $\alpha, \beta \in \mathbb{R}$,

$\alpha u + \beta v = \underline{0}$ only if $\alpha = 0, \beta = 0$

hence u, v are linearly independent

(b) $u = (1, -3)$, $v = (-2, 6)$

for any $\alpha, \beta \in \mathbb{R}$

$\alpha u + \beta v = (\alpha - 2\beta, -3\alpha + 6\beta)$

$\alpha u + \beta v = \underline{0}$

$\Rightarrow (\alpha - 2\beta, -3\alpha + 6\beta) = (0, 0)$

$\Rightarrow \alpha - 2\beta = 0$ — (i)

$-3\alpha + 6\beta = 0$ — (ii)

From $\alpha - 2\beta = 0$

$\therefore$ set $\beta = k$

$\alpha = 2k$

$(\alpha, \beta) = \{(2k, k) : k \in \mathbb{R}\}$

$(2, 1)$ is a case for such solutions;

$\Rightarrow$ there exist $\alpha, \beta \in \mathbb{R}$ such that $\alpha \neq 0, \beta \neq 0$

$\alpha u + \beta v = \underline{0}$

$\Rightarrow u, v$ are linearly dependent vectors.

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(2) Let $f, g \in C^1([a, b])$ (vector space of all real-valued continuously differentiable functions on $[a, b]$). Show that f, g are linearly dependent if and only if the Wronskian, $W(f, g) = 0$.

Solution

Suppose f, g are linearly dependent; it implies that

$f = \alpha g$ for scalar $\alpha \in \mathbb{R}$

$$W(f, g) = \begin{vmatrix} f & g \\ f' & g' \end{vmatrix} = \begin{vmatrix} \alpha g & g \\ \alpha g' & \alpha g' \end{vmatrix}$$

$= \alpha g g' - \alpha g g'$

$= 0$

Conversely, suppose $W(f, g) =$

$\Rightarrow W(f, g) = \begin{vmatrix} f & g \\ f' & g' \end{vmatrix}$

$\Rightarrow fg' - f'g = 0$

$\Rightarrow \int \frac{g'}{g} = \int \frac{f'}{f}$

$\ln g = \ln(\alpha f)$ $\alpha \in \mathbb{R}$

$\Rightarrow g = \alpha f$

$\Rightarrow f, g$ are linearly dependent.

Hence, we conclude that f, g are linearly dependent if and only if the Wronskian, $W(f, g) = 0$

3. Show that f, g, h are linearly independent

(a) $f(t) = e^t,\ g(t) = \sin t,\ h(t) = t^2$

(b) $f(t) = e^t,\ g(t) = e^{2t},\ h(t) = t$

Solution.

(b) $f(t) = e^t,\ g(t) = e^{2t},\ h(t) = t$

$$W(f,g,h) = \begin{vmatrix} f & g & h \\ f' & g' & h' \\ f'' & g'' & h'' \end{vmatrix} = \begin{vmatrix} e^t & e^{2t} & t \\ e^t & 2e^{2t} & 1 \\ e^t & 4e^{2t} & 0 \end{vmatrix}$$

$= e^t(-4e^{2t}) - e^{2t}(-te^t) + t(4e^{3t} - 2e^{3t})$

$= -3e^{3t} + 2te^{3t}$

$= e^{3t}[2t-3]$

$= 0$ only if $2t - 3 = 0$ i.e. $t = 3/2$

$\therefore f, g, h$ are linearly independent.

4. Suppose u, v, w are linearly independent vectors. Prove that S is linearly independent where

a) $S = \{u+v-2w,\ u-v-w,\ u+w\}$

b) $S = \{u+v-3w,\ u+3v-w,\ v+w\}$

c) $S = \{u+v,\ u-v,\ u-2v+w\}$

Solution.

u, v, w are linearly independent vectors, it implies that for any scalars $\alpha, \beta, \gamma \in \mathbb{R}$

$\alpha u + \beta v + \gamma w = 0$ only if $\alpha = \beta = \gamma = 0$

$S = \{u+v-2w,\ u-v-w,\ u+w\}$

For any scalars, $\lambda_1, \lambda_2, \lambda_3$; $\lambda_1 a + \lambda_2 b + \lambda_3 c$

$= \lambda_1(u+v-2w) + \lambda_2(u-v-w) + \lambda_3(u+w)$

$= (\lambda_1 + \lambda_2 + \lambda_3)u + (\lambda_1 - \lambda_2)v + (-2\lambda_1 - \lambda_2 + \lambda_3)w$

$= 0$ only if $\lambda_1 + \lambda_2 + \lambda_3 = 0$

$\lambda_1 - \lambda_2 = 0$

$-2\lambda_1 - \lambda_2 + \lambda_3 = 0$

$\Rightarrow \lambda_1 = \lambda_2 = \lambda_3 = 0$

~~a, b, c are linearly~~ For any scalars $\lambda_1, \lambda_2, \lambda_3$

$\lambda_1 a + \lambda_2 b + \lambda_3 c = 0$

only if $\lambda_1 = \lambda_2 = \lambda_3 = 0$ hence;

a, b, c are linearly independent i.e. S is a linearly independent set.

Basis and Dimension

Let V be a linear vector space over a field of scalars K. A set of vectors in V $\{v_1, v_2, \ldots, v_m\}$ is said to form a basis for V if

i) $v_1, v_2, \ldots, v_m$ are linearly independent vectors.

ii. S spans V i.e. every vector $v \in V$ can be expressed as a linear combination of $v_1, v_2, \dots, v_m$. For example, the set

$S = \{(1,0), (0,1)\}$

forms a basis for $\mathbb{R}^2$ over $\mathbb{R}$. If a ~~the~~ basis S for a linear vector space V over $\mathbb{R}$ has n-vectors then the dimension of V,

$\dim V = n$

Example,

i) Determine whether or not the following vectors form a basis for $\mathbb{R}^3$

a) $(1,1,1)$, $(1,2,3)$, $(2,-1,1)$

b) $(1,1,2)$, $(1,2,5)$, $(5,3,4)$

Solution

a) $(1,1,1)$, $(1,2,3)$, $(2,-1,1)$

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 2 & -1 & 1 \end{pmatrix} \xrightarrow[-2R_1+R_3]{-R_1+R_2} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & -3 & -1 \end{pmatrix} \xrightarrow[3R_2+R_3]{} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 5 \end{pmatrix}$$

$$\xrightarrow[\frac{1}{5}R_3]{} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

Since the elementary row operation on the matrix still has Echelon form of 3 rows, hence, the set of vectors are linearly independent. Moreover, every vector $v \in \mathbb{R}^3$ can be expressed as linear combination of $(1,1,1)$, $(1,2,3)$ and $(2,-1,1)$. Hence, they form a basis for $\mathbb{R}^3$.

Example:- Find the dimension and a basis of the subspace W of ~~P_3~~ $P_3[t]$ spanned by

$u = t^3 + 2t^2 - 3t + 4$

$v = 2t^3 + 5t^2 - 4t + 7$

$w = t^3 + 4t^2 + t + 2$

Solution

We write the coefficients of the polynomials in matrix form

$$\begin{pmatrix} 4 & -3 & 2 & 1 \\ 7 & -4 & 5 & 2 \\ 2 & 1 & 4 & 1 \end{pmatrix}$$

$$\xrightarrow{R_{13}} \begin{pmatrix} 2 & 1 & 4 & 1 \\ 7 & -4 & 5 & 2 \\ 4 & -3 & 2 & 1 \end{pmatrix} \xrightarrow[-2R_1+R_3]{\frac{1}{2}R_1} \begin{pmatrix} 1 & \frac{1}{2} & 2 & \frac{1}{2} \\ 7 & -4 & 5 & 2 \\ 0 & -5 & -6 & -1 \end{pmatrix}$$

Using another method

$$\xrightarrow[-R_1+2R_3]{-7R_1+4R_2} \begin{pmatrix} 4 & -3 & 2 & 1 \\ 0 & 5 & 6 & 1 \\ 0 & 5 & 6 & 1 \end{pmatrix} \xrightarrow[R_2-R_3]{} \begin{pmatrix} 4 & -3 & 2 & 1 \\ 0 & 5 & 6 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

A basis for subspace W spanned by those vectors =

$\{t^3 + 2t^3 - 3t + 4, t^3 + \ldots\}$

$\dim W = 2$ where 2 is the no. of non-zero rows.

Example 3.

Let U be the subspace of $\mathbb{R}^5$ generated by
$\{(1,-1,-1,-2,0), (1,-2,-2,0,-3), (1,-1,-2,-2,1)\}$
and W the subspace generated by
$\{(1,-2,-3,0,-2), (1,-1,-3,2,-4), (1,-1,-2,2,-5)\}$
find a basis and dimension for
i) $U+W$ ii) $U \cap W$

N.B.

$\dim U + \dim W = \dim(U+W) + \dim(U \cap W)$

Solution

For $U+W$, the matrix is

$$\begin{pmatrix} 1 & -1 & -1 & -2 & 0 \\ 1 & -2 & -2 & 0 & -3 \\ 1 & -1 & -2 & -2 & 1 \\ 1 & -2 & -3 & 0 & -2 \\ 1 & -1 & -3 & 2 & -4 \\ 1 & -1 & -2 & 2 & -5 \end{pmatrix}.$$

$$\rightarrow \begin{pmatrix} 1 & -1 & -1 & -2 & 0 \\ 0 & 1 & 1 & -2 & 3 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -3/2 \end{pmatrix}$$

A basis for $U+W$ is $=$
$\{(1,-1,-1,-2,0), (0,1,1,-2,3), (0,0,1,0,-1), (0,0,0,1,-3/2)\}$

$\dim(U+W) = 4$

ii. For U;

$$\begin{pmatrix} 1 & -1 & -1 & -2 & 0 \\ 1 & -2 & -2 & 0 & -3 \\ 1 & -1 & -2 & -2 & 1 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & -1 & -1 & -2 & 0 \\ 0 & 1 & 1 & -2 & 3 \\ 0 & 0 & 1 & 0 & -1 \end{pmatrix}$$

A basis for U is
$\{(1,-1,-1,-2,0), (0,1,1,-2,3), (0,0,1,0,-1)\}$

$\dim U = 3$

For W

$$\begin{pmatrix} 1 & -2 & -3 & 0 & -2 \\ 1 & -1 & -3 & 2 & -4 \\ 1 & -1 & -2 & 2 & -5 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -2 & -3 & 0 & -2 \\ 0 & 1 & 0 & 2 & -2 \\ 0 & 0 & 1 & 0 & -1 \end{pmatrix}$$

2) a basis for W is
$\{(1,-2,-3,0,-2), (0,1,0,2,-2), (0,0,1,0,-1)\}$
and $\dim W = 3$

so;

$\dim(U \cap W) = \dim U + \dim W - \dim(U+W)$
$= 3 + 3 - 4 = 2$

a basis for $U \cap W$ is
$\{(1,-2,-3,0,-2), (0,0,1,0,-1)\}$

To determine a basis for $U \cap W$;
Let $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3 \in \mathbb{R}$
$\alpha_1(1,-1,-1,-2,0) + \alpha_2(0,1,1,-2,3)$
$+\alpha_3(0,0,1,0,-1) \equiv \beta_1(1,-2,-3,0,-2)$
$+\beta_2(0,1,0,2,-2) + \beta_3(0,0,1,0,-1)$

8/10/2019 System of Linear Equations:

Let $Ax = 0$ be a homogenous system of a linear equations of n unknowns such that the rank of A is equal to r then:

(a) If $r = n$, the system has only the trivial solution; and
(b) If $r < n$, there exists $n-r$ linearly independent solution and every solution is a linear combination of them.

Examples:
(1) Find a basis over $\mathbb{R}$ for the solution space of each of the following system of linear equation.

(a)
$x+y+z=0$
$y+2z+2w=0$

(b)
$4x-y-2z-w=0$
$2x+y-z-2w=0$
$2y-4z-8w=0$
$2x-11y+2z+4w=0$

Solution:

$x+y+z=0$
$y+2z+2w=0$
Let $z = k_1$, $w = k_2$
$y = -2k_1 - 2k_2$
$x = -y - k_1$
$x = k_1 + 2k_2$, $y = -2k_1 - 2k_2$ $z = k_1$ $w = k_2$

Solution Space
$\{(w,x,y,z) : (k_2, k_1+2k_2, -2k_1-2k_2, k_1) ; k_1, k_2 \in \mathbb{R}\}$
$\{k_1(0,1,-2,1) + k_2(1,2,-2,0) : k_1, k_2 \in \mathbb{R}\}$
The basis is
$\{(0,1,-2,1), (1,2,-2,0)\}$

Determine the values of k such that the system
$x+y+kz=2$
$3x+4y+2z=k$
$2x+3y-z=1$
has; (a) Unique solution (b) infinitely many solution
(c) no solution

Solution

$x + y + kz = 2$

$3x + 4y + 2z = k$

$2x + 3y - z = 1$

The matrix of coefficients is

$$\begin{pmatrix} 1 & 1 & k \\ 3 & 4 & 2 \\ 2 & 3 & -1 \end{pmatrix}$$

for the system to have unique solution; the matrix of the coefficients must be non-singular i.e inverse of A must exist. It implies $A^{-1} \neq 0$.

So, the rows of the matrix must be linearly independent

$$\begin{pmatrix} 1 & 1 & k \\ 3 & 4 & 2 \\ 2 & 3 & -1 \end{pmatrix} \xrightarrow{-3R_1 + R_2} \begin{pmatrix} 1 & 1 & k \\ 0 & 1 & -3k+2 \\ 0 & 1 & -2k+1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & k \\ 0 & 1 & -2k+2 \\ 0 & 0 & k-3 \end{pmatrix}$$

$\Rightarrow k - 3 \neq 0, \ k \neq 3$

i. For the system to have a unique solution, $k \neq 3$

ii. The system has infinitely many solutions when $k = 3$ because of the linear dependence of rows.

iii. There is no $k \in \mathbb{R}$ such that the system has no solution

09/10/19

Linear Mappings.

1. Linear Transformation.
2. Kernel of linear Mappings
3. Image of linear Transformation
4. Rank of a linear Transformation.
5. Nullity of a linear Transformation.
6. Transition Matrix.
7. Matrix representation of linear Transformation

(C) Linear Transformation :-

Definition :- Let V, U be vector spaces over the same scalar field K. A mapping $F: V \longrightarrow U$ is called a linear transformation or Mapping if the following 2 conditions are satisfied

i) For any vectors $v, w \in V$,

$$F(v+w) = F(v) + F(w)$$

ii) For any scalar $\alpha \in K$ and vector $v \in V$;

$$F(\alpha v) = \alpha F(v)$$

Every linear mappings maps the domain to the co-domain (elements in the domain $\longrightarrow$ elements in the codomain)

a function mapping origin $\longrightarrow$ origin is linear, but Linearity doesn't always involve origin $\longrightarrow$ Origin

(i.e) the converse is not totally true

$$\mathbb{R}^2 (0,0) \neq (0,0,0) \mathbb{R}^3$$

For every solution, you must first check for the origins of the domains to that of the co-domains, then check for subsequent domains.

– Alternatively, $F: V \longrightarrow U$ is called Mapping | transformation.

if

$$F(\alpha v + \beta w) = \alpha F(v) + \beta F(w)$$

for any $\alpha, \beta \in K$ and $v, w \in V$

Examples of Linear Mappings

2. Derivative and Integral

Let V be the vector space of polynomials $P(t)$ over the real field $\mathbb{R}$ ~~xxxxxx~~

Let $D: V \longrightarrow V$ be the derivative mapping such that

$$D(u+v) = \frac{d}{dt}(u+v) = \frac{du}{dt} + \frac{dv}{dt} \text{ and } = D(u) + D(v).$$

$$D(\lambda u) = \frac{d}{dt}(\lambda u) = \lambda \frac{du}{dt} = \lambda D(u).$$

This shows that D is a linear mapping.

Example 2: $J - V \Rightarrow \mathbb{R}$ be the integral mapping defined by

$$J(f(t)) = \int_0^1 f(t)\,dt, \quad f(t) \in V$$

Show that J is a linear mapping.

Let $f(t), g(t) \in V$.

$$J(f(t) + g(t)) = \int_0^1 (f(t) \times g(t))\,dt = \int_0^1 f(t)\,dt + \int_0^1 g(t)\,dt = J(f(t)) + J(g(t))$$

ii. $$J(\lambda f(t)) = \int_0 \lambda f(t)\,dt = \lambda \int_0^1 f(t)\,dt = \lambda J(f(t))$$

Integration & differentiation are linear mappings.

3. Zero Mapping.

Example.

Let $F: V \longrightarrow U$ be the mapping that assigns the zero vector $\vec{0} \in U$ to every vector $v \in V$

Proof: Let $u, v \in V$

$F(u+v) = \vec{0}$; but $F(u) = \vec{0}$

$F(u+v) = \vec{0} = \vec{0} + \vec{0}$ $\quad F(v) = \vec{0}$

$= F(u) + F(v)$

~~Thus~~ image of sum = sum of image

Let $\lambda \in K$ and $v \in V \Rightarrow (\lambda v) \in V$

$\therefore F(\lambda v) = \vec{0}$ since F is a zero mapping.

~~Also~~ $v \in V$, $F(v) = \vec{0}$ 'cause F is a zero mapping.

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$\Rightarrow \lambda F(v) = \lambda \cdot \vec{0} = \vec{0}$.

In conclusion $F(\lambda v) = \vec{0} = \vec{0} = \lambda \vec{0} = \lambda F(v)$

ie $F(\lambda v) = \lambda F(v)$.

4. Identity Mapping

Let $I: V \to V$ be a mapping which maps $v \in V$ on to itself

i.e. $I(v) = v$ for all $v \in V$

We claim that $I: V \to V$ is a linear mapping.

To justify this claim, we follow this approach ie

Proof:-

Let $u, v \in V \Rightarrow I(u) = u$ and

$I(v) = v$

Now $I(u+v) = u + v = I(u) + I(v)$.

ii. Let $\lambda \in K$ and $v \in V$

$\Rightarrow (\lambda v) \in V$

So $I(\lambda v) = \lambda v = \lambda I(v)$.

THEOREM*:-

Let V and U be vector spaces ~~and~~ ~~let~~ over a scalar field K. Let $\{v_1, v_2, \dots, v_n\}$ be a basis of V and let $\{u_1, u_2, \dots, u_n\}$ be any vectors in U. Then there exists a unique linear mapping $F: V \to U$ such that

this theorem is used for generating linear mappings

$F(v_1) = u_1, \; F(v_2) = u_2, \; \dots, \; F(v_n) = u_n$.

~~Examples~~

Better still, this theorem states that a linear mapping is completely determined by its values at the elements of the basis of the domain vector space.

Example: Let $F: \mathbb{R}^2 \to \mathbb{R}^2$ be the linear mapping for which $F(1,2) = (2,3)$

and $F(0,1) = (1,4)$.

Find the formula for F, that is find $F(a,b)$, $a, b \in \mathbb{R}^2$.

* is $\{(0,1), (1,2)\}$ a basis of $\mathbb{R}^2$?

Yes! Since $(1,2)$ and $(0,1)$ are linearly independent *

$(a,b) = \alpha(1,2) + \beta(0,1)$

$(a,b) = (\alpha, 2\alpha + \beta)$

$\Rightarrow a = \alpha : \alpha = a$

$b = 2\alpha + \beta = 2a + \beta \therefore \beta = b - 2a$

$\therefore (a,b) = \alpha(1,2) + \beta(0,1) =$

$a(1,2) + (b-2a)(0,1)$

$F(a,b) = F\big(\alpha(1,2) + \beta(0,1)\big)$

$= \alpha F(1,2) + \beta F(0,1)$

$= \alpha(2,3) + \beta(1,4)$

$= a(2,3) + (b-2a)(1,4) = (b, 4b-5a)$

$= \begin{pmatrix} 0 & 1 \\ -5 & 4 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$ A linear mapping has a matrix (interpretation) representation

5. Matrices as Linear Mappings

Let A be any real $m \times n$ matrix. i.e.

$$A = \begin{pmatrix} a_{11} & a_{12}, & \dots, & a_{1n} \\ a_{21} & a_{22}, & \dots, & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2}, & \dots, & a_{mn} \end{pmatrix}$$

A determines a mapping

$$F_A : K^n \longrightarrow K^m$$

defined by

$$F_A(v) = Av, \ v \in K^n ; \ v = (v_1, v_2, \dots, v_n)^T$$

n → no. of columns
m is no. of rows

clearly F_A is a linear mapping.

Proof :- Let $\alpha, \beta \in K$ and $v_1, v_2 \in K^n$

Then, $\alpha v_1 + \beta w = (\alpha v_1 + \beta w_1, \alpha v_2 + \beta w_2, \dots, \alpha v_n + \beta n)$

$$F_A(\alpha v + \beta w) = A(\alpha v + \beta w) = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \begin{pmatrix} \alpha v_1 + \beta w_1 \\ \alpha v_2 + \beta w_2 \\ \alpha v_n + \beta w_n \end{pmatrix}$$

$$= \alpha \begin{pmatrix} a_{11} & a_{12} \dots & a_{1n} \\ & & \\ a_{m1} & a_{m2} \dots & a_{mn} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \dots & a_{1n} \\ \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} \dots & a_{mn} \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix}$$

$$= \alpha A(v) + \beta A(w)$$

$$= \alpha F_A(v) + \beta F_A(w)$$

so F_A is a linear mapping.

Note that the vectors in K^n and K^m are written as columns

Kernel and Image of a Linear Mapping

Definition :- $F: V \to U$ The kernel of F, written as $\ker F$ is the set of all elements in V that are mapped onto the $\vec{0}_u$ in U i.e

$$\ker F := \{v \in V : F(v) = \vec{0}_u\}$$

Image of a linear Mapping $F: V \longrightarrow U$ is the set of elements in U that are images of some element in the domain i.e

$$\operatorname{Im} F := \{u \in U : \exists\, v \in V \text{ for which } F(v) = u\}$$

Theorem 1 :- Let $F: V \longrightarrow U$ be a linear mapping. then;

- Kernel of F is a subspace of V
- Image of F is a subspace of U.

Theorem 2 :- Suppose $v_1, v_2, \dots, v_m$ is a span a vector space V. and suppose;

$$F : V \longrightarrow U \text{ is a linear mapping. Then,}$$

$F(v_1), F(v_2), \dots, F(v_m)$ spans $\operatorname{Im} F$

Proof :- Let u be an element of $\operatorname{Im} F$. $u \in \operatorname{Im} F$. $\exists\, v \in V$

$u = F(v)$

$v = \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n$ for scalars $\alpha_1, \alpha_2, \dots, \alpha_n$

$u = F(v) = F(\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n)$.

$= \alpha_1 F(v_1) + \alpha_2 F(v_2) + \dots + \alpha_n F(v_n)$

for scalars $\alpha_1, \alpha_2, \dots \alpha_n$.

$\Rightarrow F(v_1), F(v_2), \dots F(v_m)$ is a spanning set of $\operatorname{Im} F$.

The proof is complete.

~~$F: \mathbb{R}^2 \not\to \mathbb{R}^3$~~

~~$F(x, y)$~~

KERNEL AND IMAGE OF MATRIX MAPPINGS

Consider say a 3×4 matrix A and the usual basis $\{e_1, e_2, e_3, e_4\}$ of K^4 (written as column)

$$A = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \end{pmatrix}$$

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad e_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Recall that A may be viewed as a linear mapping $A: K^4 \to K^3$ where the vectors in K^4 and K^3 are viewed as columns vector. Now the usual basis vector space K^4. So their images Ae_1, Ae_2, Ae_3, Ae_4 spans the image of A. But the vectors Ae_1, Ae_2, Ae_3, Ae_4 are precisely the column of A

i.e. $Ae_1 = (a_1, b_1, c_1)^T$, $Ae_2 = (a_2, b_2, c_2)^T$, $Ae_3 = (a_3, b_3, c_3)^T$

$Ae_4 = (a_4, b_4, c_4)^T$.

Thus, the image of A is precisely the column space of A.

On the other hand, the kernel of A consists of all vector $v = \vec{O}_u$

This means that the kernel of A is the solution space of the homogenous system $AX = 0$, called the null space of A.

RANK AND NULLITY OF A LINEAR MAPPINGS

Let $F: V \to U$ be a linear mapping. The rank of F is defined to be the dimension of its image and the nullity of F is defined to be the dimension of its kernel.

Namely rank $F = \dim(\operatorname{Im} F)$

Nullity of $F = \dim(\operatorname{Ker} F)$

~~Examples~~

THEOREM :- Let V be a finite dimensional vector space and let $F: V \to U$ be a linear mapping

Proof - Let u be an element of $\operatorname{Im}F$. $u \in \operatorname{Im}F$. $\exists v \in V$

$u = F(v)$

$v = \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n$ for scalars $\alpha_1, \alpha_2, \dots, \alpha_n$

$u = F(v) = F(\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n)$.

$= \alpha_1 F(v_1) + \alpha_2 F(v_2) + \dots + \alpha_n F(v_n)$

for scalars $\alpha_1, \alpha_2, \dots \alpha_n$

$\Rightarrow F(v_1), F(v_2), \dots F(v_m)$ is a spanning set of $\operatorname{Im}F$.

The proof is complete.

$F: \mathbb{R}^2 \not\longrightarrow \mathbb{R}^3$

$F(x, y$

KERNEL AND IMAGE OF MATRIX MAPPINGS

Consider say a 3×4 matrix A and the usual basis $\{e_1, e_2, e_3, e_4\}$ of K^4 (written as column)

$$A = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \end{pmatrix}$$

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad e_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Recall that A may be viewed as a linear mapping $A: K^4 \longrightarrow K^3$ where the vectors in K^4 and K^3 are viewed as columns vector. Now the usual basis vector space K^4. So their images Ae_1, Ae_2, Ae_3, Ae_4 span the image of A. But the vectors Ae_1, Ae_2, Ae_3, Ae_4 are precisely the column of A ie

$Ae_1 = (a_1, b_1, c_1)^T$, $Ae_2 = (a_2, b_2, c_2)^T$, $Ae_3 = (a_3, b_3, c_3)^T$

$Ae_4 = (a_4, b_4, c_4)^T$.

Thus, the image of A is precisely the column space of A.
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RANK AND NULLITY OF A LINEAR MAPPINGS:

Let $F: V \to U$ be a linear mapping. The rank of F is defined to be the dimension of its image and the nullity of F is defined to be the dimension of its kernel.

Namely rank $F = \dim(\operatorname{Im}F)$

Nullity of $F = \dim(\operatorname{Ker}F)$.

~~Examples~~

THEOREM:- Let V be a finite dimensional vector space and let $F: V \longrightarrow U$ be a linear mapping.

$\dim V = \dim(\text{Ker } F) + \dim(\text{Im} F)$
$= \text{Nullity}(F) + \text{rank}(F)$

Example:-

Let $F: \mathbb{R}^4 \longrightarrow \mathbb{R}^3$ be a linear mapping defined by
$F(x, y, z, t) = (x - y + z + t,\ 2x - 2y + 3z + 4t,\ 3x - 3y + 4z + 5t)$

(a) Find a basis and the dimension of the image of F.
(b) Find a basis and the dimension of the kernel of F.

Solution

Steps.
(1) Find any vectors that spans $\mathbb{R}^4$
(2) Find their images.
(3) From the matrix M whose rows are their images vectors and row reduced

Step (1) $e_1 = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}$, $e_2 = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}$ $e_3 = \begin{pmatrix}0\\0\\1\\0\end{pmatrix}$ $e_4 = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$

Step (2) $F(e_1) = (1, 2, 3)$, $F(e_2) = (-1, -2, -3)$ $F(e_3) = (1, 3, 4)$
$F(e_4) = (1, 4, 5)$

$$M = \begin{pmatrix}1 & 2 & 3\\ -1 & -2 & -3\\ 1 & 3 & 4\\ 1 & 4 & 5\end{pmatrix} \xrightarrow[R_2 + R_3]{R_1 + R_2} \begin{pmatrix}1 & 2 & 3\\ 0 & 0 & 0\\ 0 & 1 & 1\\ 0 & 2 & 2\end{pmatrix} \longrightarrow \begin{pmatrix}1 & 2 & 3\\ 0 & 1 & 1\\ 0 & 2 & 2\end{pmatrix} \xrightarrow{2R_2 - R_3} \begin{pmatrix}1 & 2 & 3\\ 0 & 1 & 1\\ 0 & 0 & 0\end{pmatrix}$$

Thus $(1, 2, 3)$, $(0, 1, 1)$ from a basis of Im F. Thus rank of F is 2.

(b) $\text{Ker } F = \{ v \in \mathbb{R}^4 \ \ F(v) = \vec{0} \}$
$F(x, y, z, t) = \vec{0}$
$x - y + z + t = 0$, $2x - 2y + 3z + 4t = 0$, $3x - 3y + 4z + 5t = 0$

$$\Rightarrow \begin{pmatrix}1 & -1 & 1 & 1\\ 2 & -2 & 3 & 4\\ 3 & -3 & 4 & 5\end{pmatrix}\begin{pmatrix}x\\y\\z\\t\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\xrightarrow[3R_1 + R_3]{-2R_1 + R_2} \begin{pmatrix}1 & -1 & 1 & 1\\ 0 & 0 & 1 & 2\\ 0 & 0 & 0 & 0\end{pmatrix}\begin{pmatrix}x\\y\\z\\t\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\longrightarrow \begin{pmatrix}1 & -1 & 1 & 1\\ 0 & 0 & 1 & 2\end{pmatrix}\begin{pmatrix}x\\y\\z\\t\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

$x - y + z + t = 0$ Hence, $\dim(\text{Ker } F) = 2$ or Nullity$(F) = 2$
$z + 2t = 0$

(1) Set $y = 1$, $t = 0$ to obtain the solution $(-1, 1, 0, 0)$
(2) Set $y = 0$, $t = 1$ to obtain $(1, 0, -2, 4)$

Thus $(-1, 1, 0, 0)$ and $(1, 0, -2, 1)$ from a basis for Ker(F)

Step 2: Linear Mapping and Systems of Linear Equations

Let $AX = B$ denote each form of a system of n linear equations in n unknowns.

Now, the matrix A may be viewed as linear mapping

$$A : K^n \longrightarrow K^n$$

$AX \longrightarrow B$ may be noted as the real image of $B \in K^m$ under the linear mapping A.

Furthermore, the solution of the associated homogeneous equation may be viewed as the kernel of the linear mapping A:

$\dim(\text{Ker } A) + \dim(\text{Im } A) = \dim(K^n)$ i.e. $\dim(\text{Ker } A) + \text{rank } A = n$

THEOREM: The dimension of the solution space W of a homogeneous system $AX = 0$ of linear equations is

$S = n - r$, where n is the number of unknowns and r is the rank of the coefficient matrix A.

Observe that r is also the number of pivotal variables in Echelon form of $AX = 0$. So, $S = n - r$ is also the number of free variables.

SINGULAR AND NON-SINGULAR MAPPING

Let $F : V \to U$ be a linear mapping. Recall that $F(0_V) = \vec{0}_U$

F is said to be singular if the image of some non-zero vector is zero. i.e. if there exists $V \neq 0_V \in V \ni F(v) = \vec{0}_V$.

Thus, drawing from U to V is non-singular i.e. $F : U \to V$

If zero vector $(\vec{0})$ is the only vector whose range under first That is; $\text{Ker } F = \{0\}$

THEORY:

Let $F : V \to U$ be non-singular linear mapping. Then, the image of any linearly independent cells is linearly independent i.e. A non-linear mapping deserves linear independency.

TRANSITION MATRIX

Let V be a vector space over $\mathbb{R}$ and suppose

$S = \{u_1, u_2, \ldots, u_n\}$ is a basis for V

By definition, any vector $v \in V$ can be expressed as follows

$v = \sum \alpha_i u_i = \alpha_1 u_1 + \alpha_2 u_2 + \ldots + \alpha_n u_n$ for a unique choice of scalars $\alpha_1, \alpha_2, \ldots \alpha_n \in \mathbb{R}$

The scalars $\alpha_1, \alpha_2, \ldots \alpha_n$ are called the coordinates of V relative to the ordered basis S

The vector:

$\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)^T \in \mathbb{R}^n$ is called the coordinate triple of V relative to S.

Example: Consider the basis $\{(4,3), (1,-1)\}$ and $\{(1,2), (0,1)\} \in \mathbb{R}$.

SOLUTION

$$\begin{pmatrix}7\\8\end{pmatrix} = \alpha\begin{pmatrix}4\\2\end{pmatrix} + \beta\begin{pmatrix}1\\-1\end{pmatrix}$$

$7 = 4\alpha + \beta$

$8 = 2\alpha - \beta$

$15 = 6\alpha$

$\Rightarrow \alpha = 5/2$

$8 = 2(5/2) - \beta$

$\beta = 5 - 8 = -3$

$(\alpha, \beta) = (5/2, -3)$

TRANSITION MATRIX

Let $S = \{u_1, u_2, \dots, u_n\}$ and

$S' = \{v_1, v_2, \dots, v_n\}$ be any two (2) basis of a vector V. Then, the transition matrix $P_S = S'$ from $S \to S'$ is the $n \times n$ matrix which columns are coordinates of u_i in a basis S' ie $P_S \longrightarrow S' = [u, S]$

Explanation

$u_1 = P_1 v_1 + P_2 v_2 + \dots + P_n v_n$

$u_2 = P_2 v_1 + P_2 v_2 + \dots + P_n v_n$

$\Rightarrow (P_{11}, P_{12}, \dots, P_n)^T = [u_1]_{S'}$

$u_n = P_{m} v_1 + P_m v$

$$P = \begin{pmatrix} P_{11} & P_{12} & \dots & P_m \\ P_{n1} & P_{n2} & \dots & P \\ \vdots & \vdots & & \\ P & P & \dots & P \end{pmatrix}$$

$$P^T = \begin{pmatrix} P_{21} & \dots & P_{n1} \\ P_{22} & & \vdots \\ P_{nn} & \dots & P_{nn} \end{pmatrix}$$

$$P^T = \begin{pmatrix} P_{21} & \dots & P_{n1} \\ P_{22} & \dots & \vdots \\ P_{nn} & \dots & P_{nn} \end{pmatrix} = [u_1]_{S_1}, [u_2]_{S_2}, \dots [u_n]_{S_n}$$

Let V be a vector space and $S = \{v_1, v_2, \dots, v_n\}$ and $S' = \{u_1, u_2, \dots u_n\}$ be any two bases of V

$v_1 = P_{11} u_1 + P_{12} u_2 + \dots + P_{1n} u_n$

$v_2 = P_{21} u_1 + P_{22} u_2 + \dots + P_{2n} u_n$

$\vdots$

$v_n = P_{n1} u_1 + P_{22} u_2 + \dots + P_{nn} u_n$

Find the matrix representation of T wrt bases;

$S = \{(1,1,1), (1,0,1), (1,0,0)\}$.

$S' = \{(1,2), (1,1)\}$

Solution:

$$T\begin{pmatrix}1\\1\\1\end{pmatrix} = \alpha\begin{pmatrix}1\\2\end{pmatrix} + \beta\begin{pmatrix}1\\1\end{pmatrix}$$

$$T\begin{pmatrix}1\\0\\1\end{pmatrix} = \lambda\begin{pmatrix}1\\2\end{pmatrix} + \mu\begin{pmatrix}1\\1\end{pmatrix}$$

$$T\begin{pmatrix}1\\0\\0\end{pmatrix} = \gamma\begin{pmatrix}1\\2\end{pmatrix} + \rho\begin{pmatrix}1\\1\end{pmatrix}$$

$$\Rightarrow T\begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}-2\\1\end{pmatrix} = \alpha\begin{pmatrix}1\\2\end{pmatrix} + \beta\begin{pmatrix}1\\1\end{pmatrix}$$

$$\Rightarrow T\begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}3\\-5\end{pmatrix}$$

NOTE: You can choose to substitute your values of α & β in the equation.

$$T\begin{pmatrix}1\\0\\1\end{pmatrix} = \begin{pmatrix}2\\3\end{pmatrix} = \lambda\begin{pmatrix}1\\2\end{pmatrix} + \mu\begin{pmatrix}1\\1\end{pmatrix} \Rightarrow \lambda + \mu = 2\ ;\ \lambda = 1$$

$$2\lambda + \mu = -3\ ,\ \mu = 1$$

18/10/2019.

Let V be a vector space

and $S = \{v_1, v_2, \dots v_n\}$

$S' = \{u_1, u_2, \dots u_n\}$

$v_1 = P_{11}u_1 + P_{12}u_2 + \dots + P_{1n}u_n$

$v_2 = P_{21}u_1 + P_{22}u_2 + \dots + P_{2n}u_n$

$$\begin{pmatrix}v_1\\v_2\\ \vdots\\ v_n\end{pmatrix} = \begin{pmatrix}P_{11} & P_{12} & \dots & P_{1n}\\ P_{21} & P_{22} & \dots & P_{2n}\\ \vdots & \vdots & & \\ P_{n1} & P_{n2} & \dots & P_{nn}\end{pmatrix}\begin{pmatrix}u_1\\u_2\\ \vdots\\ u_n\end{pmatrix}$$

Taking transpose ; $(AB)^T = (B^T)(A^T)$ $\ast\ast$

$$(v_1, v_2, \dots v_n) = (u_1, u_2 \dots u_n)\begin{pmatrix}P_{11}\\P_{12}\\ \vdots\\ P_{1n}\end{pmatrix}\begin{pmatrix}P_{21}\\P_{22}\\ \vdots\\ P_{2n}\end{pmatrix}\cdots\begin{pmatrix}P_{n1}\\P_{n2}\\ \vdots\\ P_{nn}\end{pmatrix} =$$

$$= (u_1, u_2 \dots u_n)\left([v_1]_{s'}\ [v_2]_{s'} \dots [v_n]_{s'}\right)$$

The Matrix $P = \begin{pmatrix}P_{11} & P_{21} & \dots & P_{n1}\\ P_{1n} & P_{22} & & P_{n2}\\ \vdots & \vdots & & \vdots\\ P_{1n} & P_{2n} & & P_{nn}\end{pmatrix}$ is called the transition matrix for the basis S to S'

Example : Let $S = \{(1,2), (2,3)\}$
$S' = \{(1,1), (1,0)\}$

(a) find the transition matrix from S to S' and vice versa

Solution :

$$\begin{pmatrix}1\\2\end{pmatrix} = \alpha\begin{pmatrix}1\\1\end{pmatrix} + \beta\begin{pmatrix}1\\0\end{pmatrix} \qquad \begin{matrix}1 = \alpha+\beta & \therefore \beta = -1\\ 2 = \alpha & \alpha = 2\end{matrix}$$

$$\begin{pmatrix}2\\3\end{pmatrix} = \gamma\begin{pmatrix}1\\1\end{pmatrix} + \lambda\begin{pmatrix}1\\0\end{pmatrix} \qquad \begin{matrix}2 = \cancel{\gamma}+\lambda & \lambda = -1\\ 3 = \gamma & \gamma = 3\end{matrix}$$

$$\Rightarrow \begin{pmatrix}1\\2\end{pmatrix} = 2\begin{pmatrix}1\\1\end{pmatrix} - 1\begin{pmatrix}1\\0\end{pmatrix} \qquad \begin{bmatrix}1\\2\end{bmatrix}_{S'} = \begin{pmatrix}2\\1\end{pmatrix}$$

$$\begin{pmatrix}2\\3\end{pmatrix} = 3\begin{pmatrix}1\\1\end{pmatrix} - 1\begin{pmatrix}1\\0\end{pmatrix} \qquad \begin{bmatrix}2\\3\end{bmatrix}_{S'} = \begin{pmatrix}3\\-1\end{pmatrix}$$

$$P_{S\to S'} = \begin{pmatrix}2 & 3\\-1 & -1\end{pmatrix}$$

Proposition : The transition matrix from any basis S of K^n to the usual basis E of K^n is the matrix whose columns are precisely the basis vector of S.

$$P_{S\to E} = (u_1, u_2, \dots u_n)$$

Proof : Let $E = \{e_1, e_2, \dots, e_n\}$
and $S = \{u_1, u_2, \dots, u_n\}$

where $e_1 = \begin{pmatrix}1\\0\\ \vdots\\ 0\end{pmatrix}$ $e_2 = \begin{pmatrix}0\\1\\ \vdots\\ 0\end{pmatrix}$ $e_n = \begin{pmatrix}0\\0\\ \vdots\\ 1\end{pmatrix}$

and $u_1 = \begin{pmatrix}u_{11}\\u_{12}\\ \vdots\\ u_{1n}\end{pmatrix}$ $u_2 = \begin{pmatrix}u_{21}\\u_{22}\\ \vdots\\ u_{2n}\end{pmatrix}$ $u_n = \begin{pmatrix}u_{n1}\\u_{n2}\\ \vdots\\ u_{nn}\end{pmatrix}$

$$u_1 = u_{11}e_1 + u_{12}e_2 + \dots + u_{1n}e_n$$
$$\vdots$$
$$u_n = u_{n1}e_1 + u_{n2}e_2 + \dots + u_{nn}e_n$$
$$\Downarrow$$

$$\begin{pmatrix}u_1\\ \vdots\\ u_n\end{pmatrix} = (e_1\, e_2 \dots e_n)\begin{pmatrix}u_{11} & u_{21} & \dots & u_{n1}\\ u_{12} & u_{22} & & \vdots\\ \vdots & \vdots & & \vdots\\ u_{1n} & u_{2n} & & u_{nn}\end{pmatrix}$$

$$P_{S\to E} = \begin{pmatrix}u_{11} & u_{21} & \dots & u_{n1}\\ u_{12} & u_{22} & & \vdots\\ \vdots & \vdots & & \vdots\\ u_{1n} & u_{2n} & & u_{nn}\end{pmatrix}$$
$$\quad\; u_1 \qquad u_2 \qquad\quad u_n$$
$$= (u_1\ u_2 \dots u_n)$$

Let V be a vector space and

$S = \{v_1, v_2, \ldots v_n\}$ and

$S' = \{u_1, u_2, \ldots u_n\}$

be any two bases for V.

Suppose $v \in V$.

Let $[v]_S$ be the coordinate vector of v relative to basis S while $[v]_{S'}$ be the coordinate vector of v relative to basis S'.

$$[v]_{S'} = P_{S \to S'} [v]_S$$

$$P_{S' \to S} = P^{-1}_{S \to S'}$$

$(v_1\ v_2 \ldots v_n) P^{-1}_{S \to S'} = (u_1\ u_2 \ldots u_n) P_{S \to S'} \cdot P^{-1}_{S \to S'}$

$(u_1\ u_2 \ldots u_n) = (v_1\ v_2 \ldots v_n)(P^{-1}_{S \to S'})$

Proof: $v = \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n = (v_1\ v_2 \ldots v_n)\begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_n\end{pmatrix}$

also $v = \beta_1 u_1 + \beta_2 u_2 + \ldots + \beta_n u_n = (u_1, u_2 \ldots u_n)\begin{pmatrix}\beta_1\\ \beta_2\\ \beta_n\end{pmatrix}$

$(v_1\ v_2 \ldots v_n) = (u_1\ u_2 \ldots u_n) P_{S \to S'}$

$(v_1\ v_2 \ldots v_n)\begin{pmatrix}\alpha_1\\ \vdots\\ \alpha_n\end{pmatrix} = (u_1\ u_n \ldots u_n) P_{S \to S'}\begin{pmatrix}\alpha_1\\ \alpha_2\\ \vdots\\ \alpha_n\end{pmatrix}$

$(u_1, u_2 \ldots u_n)\begin{pmatrix}\beta_1\\ \vdots\\ \beta_n\end{pmatrix} = (u_1, u_2\ u_3 \ldots u_n) P_{S \to S'}\begin{pmatrix}\alpha_1\\ \vdots\\ \alpha_n\end{pmatrix}$

$\Rightarrow \begin{pmatrix}\beta_1\\ \vdots\\ \beta_n\end{pmatrix} = P_{S \to S'}\begin{pmatrix}\alpha_1\\ \alpha_2\\ \vdots\\ \alpha_n\end{pmatrix}$

$$[v]_{S'} = P_{S \to S'} [v]_S$$

MATRICES

Definition.

A set mn numbers, real or complex arranged in a rectangular order of m rows and n-columns such as

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

is called a matrix in $m \times n$. Each element of the matrix is called an entry.

The numbers in the array are called the uppercase letters such as $A, B \ldots$

(2) A matrix can be denoted by a representative element enclosed in the bracket such as $A = (a_{ij})$, $B = (b_{ij})$

Note (i) In case $m = n$, the rectangular array become a square and so the matrix have numbers of rows of columns equal is called a square matrix of order n.

ii. Any matrix obtained by deleting any numbers of rows & any numbers of columns formed from a given matrix is said to be a sub-matrix of the given matrix.

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}; \quad A_1 = \begin{pmatrix} a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{pmatrix}$$

$$A_2 = \begin{pmatrix} a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{pmatrix}$$

SPECIAL TYPE OF MATRICES

(1) A square matrix of order n that has 1's on the main diagonal and 0's elsewhere is called a unit matrix or identity matrix denoted by I_n. Thus,

$$I_n = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

All elements except the diagonal elements are zero

(2) A matrix having all its elements as zero is called a null matrix or zero matrix is denoted by O

Examples are

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, (0), \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

(3) A square matrix in which all the entries of the main diagonal are zero is called a diagonal matrix

$$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 6 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

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A general $m \times n$ diagonal matrix D can be written as

$$D = \begin{pmatrix} d_1 & 0 & 0 & \cdots & 0 \\ 0 & d_2 & 0 & \cdots & 0 \\ 0 & 0 & d_3 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & d_n \end{pmatrix}$$

A diagonal matrix is invertible if and only if all its diagonal entries are non-zero, the inverse of D is

$$D^{-1} = \begin{pmatrix} 1/d_1 & 0 & 0 & \cdots & 0 \\ 0 & 1/d_2 & 0 & \cdots & 0 \\ \vdots \\ 0 & 0 & 0 & \cdots & 1/d_n \end{pmatrix}$$

4. Transpose of a Matrix is formed by writing its columns as rows. E.g. if A is the $m \times n$ matrix defined by

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

Then the transpose of matrix A denoted by A^T is the $n \times m$ matrix A^T given by

$$A^T = \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{pmatrix}$$

Properties of The Transpose

If the size of matrices such that the following operations can be performed;

(i) $(A^T)^T = A$ (ii) $(A \pm B)^T = A^T \pm B^T$ (iii) $(AB)^T = (B^T A^T)$

(iv) $(kA)^T = kA^T$ where k is any scalar.

(v) If A is invertible, A^T is also invertible and $(A^T)^{-1} = (A^{-1})^T$

5. Lower and upper triangular matrices

A square matrix in which all the entries above the main diagonal are zero is called lower triangular matrix and a square matrix in which all the entries below the main diagonal are zero is called upper triangular matrix

E.g

$$\begin{pmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix} \Rightarrow \text{Lower triangular matrix}$$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{44} \end{pmatrix} \Rightarrow \text{Upper triangular matrix}$$

NB (a) Diagonal matrices are both upper and lower triangular matrices

(b) A square matrix $A = (a_{ij})$ is upper triangular iff $a_{ij} = 0 \; \forall i > j$

(c) A square matrix $A = (a_{ij})$ is a lower triangular iff $a_{ij} = 0 \; \forall i < j$

(d) The transpose of a lower triangular matrix is an upper triangular matrix and vice-versa.

(e) The product of lower triangular matrices is a lower triangular matrix. Same occurs for upper triangular matrix

(f) A triangular matrix is invertible iff its diagonal entries are non-zero.

6. Symmetrical Matrices

A square matrix A is symmetric if $A = A^T$.

If $A = (a_{ij})$ is a symmetric matrix, then $a_{ij} = a_{ji} \; \forall \; i \neq j$.

If A and B are symmetric matrices with the same size and if k is any scalar, then; A^T

i. A^T is symmetric

ii. $(A \pm B)^T$ is symmetric

iii. kB is symmetric

iv. A^{-1} is symmetric.

N-B. (i) If A is any matrix of any order, then $A^T A$ and AA^T are symmetric

ii. A square matrix A is called skew-symmetric if $A^T = -A$

iii. An orthogonal matrix is a square matrix for which

$A^T A = I = AA^T$; $A^T = A^{-1}$

iv. A non-zero square matrix A is said to be nilpotent if $A^n = 0 \; \exists \, n \in N$

i.e. $A \neq 0$; $A^1 \neq 0$, $A^2 \neq 0$, $A^3 \neq 0$, $A^{n-1} \neq 0$, $A^n = 0$

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

MTH 202 NOTE

2018/19 SESSION

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v. A square matrix is idempotent if $A^2 = A$

Equal Matrices

Two matrices $A = (a_{ij})$ & $B = (b_{ij})$ are equal if they have the same size $m \times n$ and $a_{ij} = b_{ij}$ for $1 \leq i \leq m$ and $1 \leq j \leq n$.

Example:

i) if $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}$ & $B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{pmatrix}$ are equal if

$a_{11} = b_{11}$, $a_{12} = b_{12}$, ..., $a_{32} = b_{32}$

ii.

iii. The matrices $\begin{pmatrix} 2 & 3 \\ 0 & 1 \\ 5 & 4 \end{pmatrix}$ & $\begin{pmatrix} 2 & 3 \\ 6 & 1 \\ 5 & 4 \end{pmatrix}$ are comparable because they have the same order but not equal as $0 \neq 6$

iv. The matrices $\begin{pmatrix} 3 & 2 & 5 \\ 0 & 9 & 3 \end{pmatrix}$ & $\begin{pmatrix} 3 & 2 & 5 \\ 0 & 9 & 3 \end{pmatrix}$ are equal

because the two conditions of equality are satisfied.

Equivalent Relation Of Matrices

If there are matrices A, B and C of the same size $m \times n$ such that

i. $A = A$ (Reflexive)

ii. $A = B$ implies $B = A$ is Symmetric

iii. if $A = B$ and $B = C$ then $A = C$ is transitive.

Def II

The elements a_{ij}; $(i = j)$ of a square matrix A are called its diagonal element and their sum as trace of A denoted by $tr.A = \sum_{i=1}^{n} a_{ii}$

$$tr.A = a_{11} + a_{22} + \cdots + a_{nn}$$

$$= \sum_{i=1}^{n} a_{ii}$$

Matrix Addition

You can add two matrices (of the same size) by adding corresponding entries.

Definition

If $A = (a_{ij})$ and $B = (b_{ij})$ are matrices of size $n \times m$, their sum is the $n \times m$ matrix given by

$$A + B = (a_{ij}) + (b_{ij})$$
$$= (a_{ij} + b_{ij})$$

The sum of two matrices with different sizes is undefined

i.e. If $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$ and

$B = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{pmatrix}$

then,

$A + B = \begin{pmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \cdots & a_{1m} + b_{1n} \\ a_{21} + b_{21} & a_{22} + b_{22} & \cdots & a_{2m} + b_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} + b_{m1} & a_{m2} + b_{m2} & & a_{mn} + b_{mn} \end{pmatrix}$

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NOTE:

i. $tr.(A+B) = tr.(A) + tr.(B)$

ii. if $A = (a_{ij})$ and $B = (b_{ij})$ are matrices of the same order, then $A - B = (a_{ij} - b_{ij})$

SCALAR MULTIPLICATION OF MATRICES

Real numbers are referred to as scalars

Definition:

If $A = (a_{ij})$ is $m \times n$ matrix and $c \in \mathbb{R}$ is a scalar, then the scalar multiple of A by c is the $m \times n$ matrix given by $cA = c(a_{ij})$

NOTE:-

(i).

be a finite no of matrices each of order $m \times p$ (say) and $\alpha_1, \alpha_2, \dots, \alpha_n$ are scalars then $\alpha_1 A_1 + \alpha_2 A_2 + \dots + \alpha_n A_n = \alpha_1 a_{ij}^1 + \alpha_2 a_{ij}^2 + \dots + \alpha_n a_{ij}^n$

$$\sum_{k=1}^{n} \alpha_k A_k = \sum_{k=1}^{n} \alpha_k a_{ij}^k$$

1. $tr.\ \lambda A = \lambda\ tr.A$

Properties Of Matrix Addition

1. Commutative Law.

If A and B are two matrices of the same size say $m \times n$ then $A+B = B+A$.

2. Associative Law.

If A, B and C are three matrices of the same order then $A+(B+C) = (A+B)+C$

3. Distributive Law.

If A and B are matrices of the same order and k is a scalar, then $k(A+B) = kA + kB$.

4. Existence of Additive Identity

If A is a matrix say $(m \times n)$ & O (null matrix) of the same order, then,

$$A + O = A = O + A$$

O is said to be an additive identity of A.

5. Existence of Additive Inverse / Negative Matrix.

Let A be a matrix of any order (let say $m \times n$), there exist a matrix $-A$ of the same order $m \times n$ such that

$A + (-A) = A - A = 0$

$-A$ is said to be an additive inverse of A

(6) The cancellation laws:

If A, B, C are matrices conformable for addition, then the relation

$$A + B = A + C \quad \text{iff } B = C$$

$$A + B = A + C$$

$$A + B - A = C$$

$$B = C$$

Problems.

i- If $A = \begin{pmatrix} 2 & 1 & -3 \\ 0 & 3 & 2 \\ -1 & 1 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 3 & 2 & -1 \\ 4 & 5 & 2 \\ 2 & 3 & 0 \end{pmatrix}$ $C = \begin{pmatrix} 4 & 2 & 1 \\ 0 & 2 & 3 \\ 1 & 3 & -2 \end{pmatrix}$

verify that;

i. $A + B = B + A$

ii. $A + (B - C) = (A + B) - C$

iii. Determine the matrix D such that

$A + D = B \Rightarrow A = B - D \quad A - B = D$

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MULTIPLICATION OF MATRICES

Definition :- Two matrices $A = (a_{ij})$ and $B = (b_{ij})$ of order $m \times n$ and $n \times p$ respectively are conformable for multiplication if and only if the number of columns in A is equal to the numbers of rows in B.

$$\underbrace{A = m \times \underline{n} \qquad B = \underline{n} \times p}_{\text{equal}}$$

AB is an $m \times p$ matrix

$$AB = (c_{ij}) = \sum_{k=1}^{n} a_{ik}b_{kj} = a_{i1}b_{1j} + a_{i2}b_{2j}, \dots, a_{in}b_{nj}$$

i. Let $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ $B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{pmatrix}$

3×3 equal 3×2

order

$$AB = \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} & a_{21}b_{12} + a_{22}b_{22} + a_{33}b_{32} \\ a_{31}b_{11} + a_{32}b_{21} + a_{33}b_{31} & a_{31}b_{12} + a_{32}b_{22} + a_{33}b_{32} \end{pmatrix}$$

ii. Find the product AB where

$A = \begin{pmatrix} -1 & 3 \\ 4 & -2 \\ 5 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} -3 & 2 \\ -4 & 1 \end{pmatrix}$

A B

3 X 2 $\underbrace{\quad}_{\text{equal } \underset{\sigma}{\text{is}} \text{ conformable}}$ 2 X 2

$\boxed{\text{order of the product}}$

$$AB = \begin{pmatrix} 3+(-12) & -2+(3) \\ -12+(8) & 8+(-2) \\ -15+0 & 10+0 \end{pmatrix} = \begin{pmatrix} -9 & 1 \\ -4 & 6 \\ -15 & 10 \end{pmatrix}$$

The general pattern for matrix multiplication is as follows.
To obtain the elements in the ith row and jth column; use the ith row of A and the jth column of B.

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{i1} & a_{i2} & a_{i3} & \cdots & a_{in} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & \cancel{b_{13}}\, b_{13} & \cdots & b_{1j} & \cdots & b_{1p} \\ b_{21} & b_{22} & b_{23} & & b_{2j} & \cdots & b_{2p} \\ \vdots & \vdots & \vdots & & \vdots & & \vdots \\ b_{n1} & b_{n2} & b_{n3} & & b_{nj} & & b_{np} \end{pmatrix}$$

$$= \begin{pmatrix} C_{11} & C_{12} & \cdots & C_{1j} & \cdots & C_{1p} \\ C_{21} & C_{22} & \cdots & C_{2j} & & C_{2p} \\ C_{i1} & C_{i2} & \cdots & C_{ij} & & C_{ip} \\ C_{n1} & C_{n2} & \cdots & C_{nj} & & C_{np} \end{pmatrix}$$

where a_{ij} is defined by eqn 1.1

NOTE:

i. If the product AB, the matrix A is known as pre-factor and B as post-factors.

ii. $tr.(AB) = tr(BA)$;

all matrices of square order

PROPERTIES OF MATRIX MULTIPLICATION.

Let A, B, C be matrices of suitable orders such that matrices multiplication are conformable, then;

i. $AB \neq BA$; The commutative law for matrix multiplication does not hold in general.

ii. $A(B+C) = AB + AC$; The distributive law holds.

iii. $A(BC) = (AB)C$. The associative law holds.

iv. If A be a matrix of order $m \times n$ and O is a null-matrix of ordera $m \times p$ i.e. $AO = O$

Also let O be $m \times n$ null matrix and A be a matrix of order $m \times p$ then the product is a null matrix of order $n \times p$ i.e. $OA = O$ in conclusion. If A and O are square matrices of order m,

$$AO = OA = O$$

v. If the product of 2 matrices A & B is a null matrix; then it is not essential that either of them is a null matrix. e.g.

$A = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$

Then;

$$AB = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

vi. Let I be an m row unit matrix and A be an $m \times n$ matrix. Then, $I \cdot A = A$

Example;

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad A = \begin{pmatrix} 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 \\ 3 & 4 & 2 & 0 \end{pmatrix}$$

vii. If A is a square matrix of order m, then

$$A^2 = A \cdot A$$

$$A^3 = A^2 \cdot A = (A \cdot A)A = A(A \cdot A)$$

In general, $A^p \cdot A^q = A^{p+q}$

$$(A^p)^q = A^{pq}$$

Corollary:

If I is a unit matrix of any order, show that

$$I^2 = I^3 = \cdots I^p = I$$

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30/10/19.

The Conjugate Of A Matrix

If the elements/entries of a matrix A are complex quantities, then the matrix obtained from A by replacing its element by the corresponding conjugate number is said to be conjugate matrix of A and it is denoted by $\bar{A}$ or A^*. Thus, if $A = (a_{ij})$ then $\bar{A} = (\bar{a}_{ij})$ where $\bar{a}_{ij}$ denotes the conjugate of a_{ij} e.g. If A;

$$= \begin{pmatrix} 1-2i & -3+5i & 5 & 3i \\ -4+3i & 4-5i & -2i & i \\ -2-7i & 2+3i & 0 & -2 \end{pmatrix}$$

To get the conjugate of matrix A,

$$\bar{A} = \begin{pmatrix} 1+2i & -3-5i & 5 & -3i \\ -4-3i & 4+5i & 2i & -i \\ -2+7i & 2-3i & 0 & -2 \end{pmatrix}$$

Properties Of Conjugate Matrices

i. The conjugate of the conjugate of matrix A coincides with itself i.e. $(\bar{\bar{A}}) = A$ e.g. $A = \begin{pmatrix} -2i & 3+4i \\ 3i & -5i \end{pmatrix}$ $\bar{A} = \begin{pmatrix} 2i & 3-4i \\ -3i & 5i \end{pmatrix}$ $\bar{\bar{A}} = \begin{pmatrix} -2i & 3+4i \\ 3i & -5i \end{pmatrix} = A$

ii. The conjugate of the sum of 2 matrices A and B (conformable for addition) is the sum of their conjugates i.e. $(\overline{A+B}) = \bar{A} + \bar{B}$

iii. If α is a complex no and A a matrix of order $m \times n$, then
$\overline{\alpha A} = \bar{\alpha}\bar{A}$

iv. The conjugate of the product of two matrices A & B (conformable for multiplication) is the product of their conjugate i.e $\overline{AB} = \bar{A}\bar{B}$

The Conjugate Transpose or Hermitian Conjugate of Matrix

The matrix which is the conjugate of a transpose of a matrix A is said to be conjugate transpose of A and is denoted by A^{θ} or $\bar{A}^{T}$ or $A^{\dagger}$ → dagger

e.g. $A = \begin{pmatrix} -2+3i & 3-4i & 2i \\ 5i & -5-2i & 3-2i \end{pmatrix}$. $A^{T} = \begin{pmatrix} -2+3i & 5i \\ 3-4i & -5-2i \\ 2i & 3-2i \end{pmatrix}$ $A^{\dagger}/\bar{A}^{T} = \begin{pmatrix} -2-3i & -5i \\ 3+4i & -5+2i \\ -2i & 3+2i \end{pmatrix}$

Properties of Conjugate Transpose Matrices.

i. The conjugate transpose of a conjugate transpose coincides with A i.e. $(A^{\theta})^{\theta} = A$.

ii. $(A+B)^{\theta} = A^{\theta} + B^{\theta}$ where A & B are conformable for addition.

iii. $(\alpha A)^{\theta} = \bar{\alpha} A^{\theta}$ where α is a complex number.

iv. $(AB)^{\theta} = B^{\theta} A^{\theta}$ where A & B are conformable for multiplication.

Def. I: A square matrix A equals a_{ij} is said to be hermitiant if A coincides with its conjugate transpose i.e. $A = A^{\theta}$

e.g $A = \begin{pmatrix} 5 & 2+3i & -i \\ 2-3i & 3 & -3-4i \\ i & -3+4i & 0 \end{pmatrix}$; $A^{T} = \begin{pmatrix} 5 & 2-3i & i \\ 2+3i & 3 & -3+4i \\ -i & -3-4i & 0 \end{pmatrix}$

$A^{\theta} = \begin{pmatrix} 5 & 2+3i & -i \\ 2-3i & 3 & -3-4i \\ i & -3+4i & 0 \end{pmatrix} = A$.

II. A square matrix $A = a_{ij}$ is said to be skew hermitant if $A^{\theta} = -A$ or $A^{T} = -\bar{A}$ e.g. $A = \begin{pmatrix} 3i & -3+4i & +4-5i \\ 3+4i & -4i & 5 \\ -4-5i & -5 & 0 \end{pmatrix}$

Problem:- Every square matrix A can be uniquely expressed as the sum of a hermitiant matrix and a skew hermitian matrix

Solution:

$A = \frac{1}{2}(A + A^\theta) + \frac{1}{2}(A - A^\theta) = P + Q$.

$P = \frac{1}{2}(A + A^\theta)$ & $Q = \frac{1}{2}(A - A^\theta)$

$P^\theta = \frac{1}{2}(A + A^\theta)^\theta =$

$P^\theta = \frac{1}{2}(A^\theta + (A^\theta)^\theta) =$

$P^\theta = \frac{1}{2}(A^\theta + A) =$

$\frac{1}{2}(A + A^\theta) = P$

{which shows that P is a hermitiant matrix}

$Q^\theta = \frac{1}{2}(A - A^\theta)^\theta$

$Q^\theta = \frac{1}{2}(A^\theta - (A^\theta)^\theta)$

$Q^\theta = \frac{1}{2}(A^\theta - A)$

$= -\frac{1}{2}(A - A^\theta) = -Q$

{which shows that Q is a hermitiant skew matrix}

Definitions:-

i. Unitary Matrix: A square matrix A having its entries as complex numbers is said to be unitary iff $A^T = \bar{A}^{-1}$ or $(\bar{A})^T = A^{-1}$ or $A^\theta A = I$. Example

If $A = \begin{pmatrix} 1/\sqrt{2} & i/\sqrt{2} \\ -i/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}$

ii. Normal Matrix: A square matrix A is said to be normal if $A^\theta A = A A^\theta$

A normal matrix includes diagonal matrix, real symmetric matrix, real skew symmetric, orthogonal matrix, hermitiant, skew hermitiant, unitary matrices

$A = \begin{pmatrix} \frac{1+i}{2} & -\frac{1+i}{2} \\ \frac{1+i}{2} & \frac{1-i}{2} \end{pmatrix}$ show it is a normal matrix

31/10/2019

RANK OF A MATRIX

Definition :- The dimension of the row (columns) space of a matrix A is called the rank of A and is denoted by Rank(A).

Example :- Find the rank of the matrix A

$$A = \begin{pmatrix} 1 & -2 & 0 & 1 \\ 2 & 1 & 5 & -3 \\ 0 & 1 & 3 & 5 \end{pmatrix}$$

Always make sure the first entry of the first row is always 1.

$-2R_1 + R_2 \longrightarrow R_2$

$$A_1 = \begin{pmatrix} 1 & -2 & 0 & 1 \\ 0 & 5 & 5 & -5 \\ 0 & 1 & 3 & 5 \end{pmatrix} \quad \frac{1}{5}R_2 \longrightarrow R_2$$

$$A_2 = \begin{pmatrix} 1 & -2 & 0 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 1 & 3 & 5 \end{pmatrix} \quad -R_2 + R_3 \longrightarrow R_3$$

$$A_3 = \begin{pmatrix} 1 & -2 & 0 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 2 & 6 \end{pmatrix} \quad \tfrac{1}{2}R_3 \longrightarrow R_3$$

$$A_4 = \begin{pmatrix} 1 & -2 & 0 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 3 \end{pmatrix}$$

Because A_4 has 3 non-zero rows, the rank of A is 3.

NULL SPACE OF A MATRIX

Consider the homogenous linear system

$\underline{A}\underline{x} = \underline{0}$ ——— (1.2)

where A is an $n \times n$ matrix.

$\underline{x} = (x_1, x_2 \ldots x_n)^T$

$$\underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$ is the column vector of the unknowns and;

$$\underline{0} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$ is the 0 vectors in $\mathbb{R}^n$

Equation 1.2 in matrix form is

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{31} & a_{32} & \cdots & a_{3n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

THEOREM:- if A is an $m \times n$ matrix, then the set of all solutions of the homogenous system of linear equation 1-2 is a subspace of

$\mathbb{R}^n$ called nullspace of A and is denoted by $N(A)$

$N(A) = \{\underline{x} \in \mathbb{R}^n \; A\underline{x} = \underline{0}\}$

The dimension of the null space of A is called nullity of A

Remark:- The nullspace of A is also called the solution space

Example:- Find the nullspace of the matrix A

$A\underline{x} = 0$

$$A = \begin{pmatrix} 1 & 2 & -2 & 1 \\ 3 & 6 & -5 & 4 \\ 1 & 2 & 0 & 3 \end{pmatrix} \quad \begin{matrix} -3R_1 + R_2 \longrightarrow R_2 . \\ -R_1 + R_3 \longrightarrow R_3 . \end{matrix}$$

$$A_1 = \begin{pmatrix} 1 & 2 & -2 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 2 & 2 \end{pmatrix} \quad -2R_2 + R_3 \longrightarrow R_3$$

$$A_2 = \begin{pmatrix} 1 & 2 & -2 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

*where 1 is the leading entry in the columns of the matrix, the number of leading 1 entry is the nullity of the matrix

$$\begin{pmatrix} 1 & 2 & -2 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$x_1 + 2x_2 - 2x_3 + x_4 = 0$

$x_3 + x_4 = 0$

$x_3 = -x_4$; let $x_4 = a$

$x_3 = -a$ $\therefore$ $x_1 + 2x_2 + 2a + a$

$x_1 = -3a - 2x_2$ Let $x_2 = b$

$x_1 = -3a - 2b$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -3a-2b \\ b \\ -a \\ a \end{pmatrix} = a\begin{pmatrix} -3 \\ 0 \\ -1 \\ 1 \end{pmatrix} + b\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

A basis for the null space of A consists of vectors

$$\begin{pmatrix} -3 \\ 0 \\ -1 \\ 1 \end{pmatrix} \text{ and } \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

These 2 vectors are the solution of $A\underline{x} = 0$

NOTE:- THEOREM:-

If A is an $m \times n$ matrix of rank 'r' then the dimension of the solution space of $A\underline{x} = \underline{0}$ is $n - r$

$n = \text{rank}(A) + \text{nullity}(A)$

Problem: Find the Rank and Nullity of A

$$A = \begin{pmatrix} 1 & 0 & -2 & 1 & 0 \\ 0 & -1 & -3 & 1 & 3 \\ -2 & -1 & 1 & -1 & 3 \\ 0 & 3 & 9 & 0 & -12 \end{pmatrix}$$

$r(A) = 3$ nullity $= 2$.

1/11/2019 THE DETERMINANTS OF A MATRIX

A determinant of a 2×2 matrix A

$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{13} & a_{14} \end{pmatrix}$ is denoted by

$\det(A)$ or $|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{13} & a_{14} \end{vmatrix} = a_{11}a_{14} - a_{12}a_{13}$.

$B = \begin{pmatrix} 3 & 7 \\ 0 & 1 \end{pmatrix}$

$|B| = \begin{vmatrix} 3 & 7 \\ 0 & 1 \end{vmatrix} = (3\times1) - (7\times0) =$

$3 - 0 = 3$

$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

Definition :- Minors and Cofactors.

If A is a square matrix, then the minor M_{ij} of the element A_{ij} is the determinant of the matrix obtained by deleting the ith row and jth column of A. The cofactor is given by

$$C_{ij} = (-1)^{i+j} M_{ij}$$

For a 3×3 matrix, the minor and co-factors are as follows

Minor of a_{11} is $M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$

Minor of a_{12} is $M_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$

Minor of a_{13} is $M_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$

Minor of a_{33} is $M_{33} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$

$C_{ij} = (-1)^{i+j} M_{ij}$

$C_{11} = (-1)^2 M_{11} = M_{11}$ $\quad C_{21} = (-1)^3 M_{21} = -M_{21}$ $\quad C_{31} = (-1)^4 M_{31} = M_{31}$

$C_{12} = (-1)^3 M_{12} = -M_{12}$ $\quad C_{22} = (-1)^4 M_{22} = M_{22}$ $\quad C_{32} = (-1)^5 M_{32} = -M_{32}$

$C_{13} = (-1)^4 M_{13} = M_{13}$. $\quad C_{23} = (-1)^5 M_{23} = -M_{23}$ $\quad C_{33} = (-1)^6 M_{33} = M_{33}$

Example :- Find the minor & cofactor of a matrix $A = \begin{pmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & 0 & 1 \end{pmatrix}$

$M_{11} = \begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix} = -1-(0) = -1$ $M_{12} = \begin{vmatrix} 3 & 2 \\ 4 & 1 \end{vmatrix} = 3-8 = -5$

$M_{13} \begin{vmatrix} 3 & -1 \\ 4 & 0 \end{vmatrix} = 4$ $M_{21} = \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} = 2$ $M_{22} = \begin{vmatrix} 0 & 1 \\ 4 & 1 \end{vmatrix} = -4$

$M_{23} = \begin{vmatrix} 0 & 2 \\ 4 & 0 \end{vmatrix} = -8$ $M_{31} = \begin{vmatrix} 2 & 1 \\ -1 & 2 \end{vmatrix} = 5$ $M_{32} = \begin{vmatrix} 0 & 1 \\ 3 & 2 \end{vmatrix} = -3$

$M_{33} = \begin{vmatrix} 0 & 2 \\ 3 & -1 \end{vmatrix} = -6$

$C_{11} = -1$ $C_{12} = 5$ $C_{13} = 4$ $C_{21} = -2$ $C_{22} = -4$

$C_{23} = 8$ $C_{31} = 5$ $C_{32} = 3$ $C_{33} = -6$

Definition: If A is a square matrix (of order 2 >) then the determinant of A is the sum of the entries in the first row of A multiplied by their co-factors. i.e

$$\det(A) = |A| = \sum_{j=1}^{n} a_{ij}C_{ij} = a_{11}C_{11} + a_{12}C_{12} + \dots + a_{1n}C_{1n} \quad \text{—(1.3)}$$

Example: Find the determinant of A

$$A = \begin{pmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & 0 & 1 \end{pmatrix}$$

Solution

$\det(A) = 0 \times (-1) + 2 \times 5 + 1 \times 4 = 14$

using a_{11}, a_{21} & a_{31}

or $\det(A) = 0\times(-1) + 3\times(-2) + 4\times(5) = 14$

Definition: Let A be a square matrix of order n. then the $\det(A)$ is given by

$$\det(A) = \sum_{i=1}^{n} a_{ij}C_{ij} = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj} \quad (-)$$

$$\det(A) = \sum_{j=1}^{n} a_{ij}C_{ij} = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in} \quad (1)$$

Find the determinant of $A = \begin{pmatrix} 1 & -2 & 3 & 0 \\ -1 & 1 & 0 & 2 \\ 0 & 2 & 0 & 3 \\ 3 & 4 & 0 & -2 \end{pmatrix}$

Alternative Method:

Consider

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$$\begin{array}{ccccc} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{array}$$

$\det A = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}$
$- a_{13}a_{22}a_{31}$

Properties of Determinants

i) If A is a triangular matrix of order n, then its determinant is the product of the entries on the main diagonal i.e.

$\det(A) = a_{11}a_{22}a_{33} \dots a_{nn}$

ii. Let A and B be square matrices (a) if B is obtained from A by interchanging 2 rows of A, then $\det(B) = -\det A$.

b. If B is obtained by adding a multiple of a row of A to another row of A, then $\det(B) = \det(A)$.

c. If B is obtained from A by multiplying a row of A by a non-zero constant c, then $\det B = c \det(A)$

iii. If A is a square matrix and any one of the following conditions is met, then $\det(A) = 0$ and A is called a singular matrix

a. An entire row (or column) consists of zeros. e.g. $A = \begin{pmatrix} 0 & 0 \\ 7 & 20 \end{pmatrix}$

b. ~~An entire row (or column)~~ Two rows (or columns) are equal.
c. One row (or column) is a multiple of another row (or column).

iv. If A and B are square matrices of order n, then;
$\det(AB) = \det(A) \cdot \det(B)$

v. If A is an nxn matrix and c is a scalar, then;
$\det(cA) = c^n \det(A)$.

vi. A square matrix A is invertible and non-singular if $\det(A) \neq 0$

vii) If A is a square matrix, then $\det(A) = \det(A^T)$

CRAMMER'S RULE

Consider a general system involving two linear equations in two variables.

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 = b_1 \\ a_{21}x_1 + a_{22}x_2 = b_2 \end{array}\right\} (1.4)$$

In matrix from

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{22} & a_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \quad (1.5)$$

Crammer's rule states:

$$x_1 = \frac{|A_1|}{|A|}, \quad x_2 = \frac{|A_2|}{|A|}$$

where $A_1 = \begin{pmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{pmatrix}$, $A_2 = \begin{pmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{pmatrix}$ & $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$

If we have

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

Then $Ax = b$ i.e.

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

and $x_1 = \frac{|A_1|}{|A|}$, $x_2 = \frac{|A_2|}{|A|}$, $x_3 = \frac{|A_3|}{|A|}$

$A_1 = \begin{pmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{pmatrix}$, $A_2 = \begin{pmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{pmatrix}$,

$A_3 = \begin{pmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{pmatrix}$

07/11/2019 # INVERSE OF NON-SINGULAR MATRICES

Singular $\det(A) = 0 \Rightarrow$ no inverse

Non-singular $\det(A) \neq 0$

The inverse of a non-singular matrix (a square matrix) A denoted by A^{-1} is given by $A^{-1} = \frac{1}{|A|}$ Adjoint(A).

Adjoint is the matrix of co-factors of A.

Cofactor is the matrix of the signed minors.

$AA^{-1} = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} 2\times2 \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} 3\times3$

E.g. Find the inverse of A; $A = \begin{bmatrix} -1 & 1 \\ -2 & 0 \end{bmatrix}$

Solution

$|A| = \begin{vmatrix} -1 & 1 \\ -2 & 0 \end{vmatrix} = 0-(-2) = 2$

$|A| \neq 0$, $\therefore$ A is non-singular and has an inverse.

Now, let $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be the inverse of $\begin{bmatrix} -1 & 1 \\ -2 & 0 \end{bmatrix}$

Recall that: $AA^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\therefore \begin{bmatrix} -1 & 1 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\begin{bmatrix} -a+c & -b+d \\ -2a & -2b \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\Rightarrow -a+c = 1 \quad \ldots \text{(i)}$

$-b+d = 0 \quad \ldots \text{(ii)}$

$-2a = 0 \Rightarrow a = 0$

$-2b = 1 \Rightarrow b = -1/2$

Using $a = 0$ in (i) ; $c = 1$ using $b = -1/2$ in eq ii

$-(-1/2) + d = 0 \therefore d = -1/2$.

$$\therefore \begin{bmatrix} -1 & 1 \\ -2 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & -1/2 \\ 1 & -1/2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 0 & -1 \\ 2 & -1 \end{bmatrix}$$

Flip the diagonal and change the signs of the other two entries to get the adjoint of a 2x2 matrix.

Find the inverse if it exists of $B \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}$

Solution:

$|B| = \begin{vmatrix} 1 & 0 \\ 2 & 3 \end{vmatrix} = 3 \quad |B| \neq 0 \therefore$ the inverse exists.

$$B^{-1} = \frac{1}{3}\begin{bmatrix} 3 & 0 \\ -2 & 1 \end{bmatrix}$$

$$B^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}, \text{ given } B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Exercise

Find the inverse of the following 2x2 matrices

1. $\begin{bmatrix} 1 & 3 \\ 5 & -2 \end{bmatrix}$ 2. $\begin{bmatrix} 4 & -1 \\ -3 & 2 \end{bmatrix}$ 3. $\begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$ 4. $\begin{bmatrix} 4 & -3 \\ 1 & 2 \end{bmatrix}$

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6/11/2019. Inverse of **nxn** non-singular matrix for $n \geq 3$

For a matrix A, non-singular; A^{-1} is given by

$$A^{-1} = \frac{1}{|A|} \operatorname{adj} A.$$

Example:- Obtain the inverse of the matrix A given by

$$A = \begin{bmatrix} 2 & -4 & 5 \\ -3 & 5 & 7 \\ 5 & 3 & -8 \end{bmatrix}$$

using the determinant method and hence obtain the solution of the linear set of equations

$$2x_1 - 4x_2 + 5x_3 = 36$$
$$-3x_1 + 5x_2 + 7x_3 = 7$$
$$5x_1 + 3x_2 - 8x_3 = -31$$

$$A = \begin{array}{c} \begin{matrix} + & \quad - & \quad + \end{matrix} \\ \begin{bmatrix} 2 & -4 & 5 \\ -3 & 5 & 7 \\ 5 & 3 & -8 \end{bmatrix} \end{array} = 2(-61) + 4(-11) + 5(-14)$$
$$= -336.$$

$M_{11} = \begin{vmatrix} 5 & 7 \\ 3 & -8 \end{vmatrix} = -61$ $\quad M_{12} = \begin{vmatrix} -3 & 7 \\ 5 & -8 \end{vmatrix} = -11$ $\quad M_{13} = \begin{vmatrix} -3 & 5 \\ 5 & 3 \end{vmatrix} = -34$

$M_{21} = \begin{vmatrix} -4 & 5 \\ 3 & -8 \end{vmatrix} = 17$ $\quad M_{22} = \begin{vmatrix} 2 & 3 \\ 5 & -8 \end{vmatrix} = -41$ $\quad M_{23} = \begin{vmatrix} 2 & -4 \\ 5 & 3 \end{vmatrix} = 26$

$M_{31} = \begin{vmatrix} -4 & 5 \\ 5 & 7 \end{vmatrix} = -53$ $\quad M_{32} = \begin{vmatrix} 2 & 5 \\ 3 & 7 \end{vmatrix} = 29$ $\quad M_{33} = \begin{vmatrix} 2 & -4 \\ -3 & 5 \end{vmatrix} = -2$

Matrix of the minors of

$$\begin{pmatrix} -61 & -11 & -34 \\ 17 & -41 & 26 \\ -53 & 29 & -2 \end{pmatrix}$$

Matrix of the signed minor i.e Cofactor of

$$\begin{pmatrix} -61 & 11 & -34 \\ -17 & -41 & -26 \\ -53 & -29 & -2 \end{pmatrix}$$

$$C^T = \begin{pmatrix} -61 & -17 & -53 \\ 11 & -41 & -29 \\ -34 & -26 & -2 \end{pmatrix} = \text{Adj } A$$

$$\therefore A^{-1} = \frac{1}{|A|} \text{Adj } A$$

$$= \frac{1}{-336} \begin{pmatrix} -61 & -17 & -53 \\ 11 & -41 & -29 \\ -34 & -26 & -2 \end{pmatrix} = \begin{pmatrix} 61/336 & 17/336 & 53/336 \\ 11/-336 & 41/336 & 29/336 \\ 34/336 & 26/336 & 2/336 \end{pmatrix}$$

(b) $$\underset{A}{\begin{pmatrix} 2 & -4 & 5 \\ -3 & 5 & 7 \\ 5 & 3 & -8 \end{pmatrix}} \underset{X}{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}} = \underset{B}{\begin{pmatrix} 36 \\ 7 \\ -31 \end{pmatrix}}$$

$$A^{-1}AX = A^{-1}B$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 61/336 & 17/336 & 53/336 \\ -11/336 & 41/336 & 29/336 \\ 34/336 & 26/336 & 2/336 \end{pmatrix} \begin{pmatrix} 36 \\ 7 \\ -31 \end{pmatrix}$$

$$\begin{pmatrix} 61/336 \times 36 + 17/336 \times 7 + 53/336 \times -31 \\ -11/336 \times 36 + 41/336 \times 7 + 29/336 \times -31 \\ 34/336 \times 36 + 26/336 \times 7 + 2/336 \times -31 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix}$$

$$x_1 = 2, \quad x_2 = 3, \quad x_3 = 4.$$

Find the inverse of the following

(1) $\begin{pmatrix} 1 & 3 & 2 \\ 4 & 1 & 3 \\ 2 & 2 & 0 \end{pmatrix}$ (2) $\begin{pmatrix} 1 & 0 & 2 \\ 1 & 3 & 4 \\ 0 & 0 & 0 \end{pmatrix}$

Solve using inverse method

a. $6x + 5y = 2$

$x + 2y = 3$

b. $x + 2y + 2z = 5$

$3x - 2y + z = -6$

$2x + y - z = -1$

07/11/2019 Eigen Values & Eigen Vectors.

Definition: Let A be an $n \times n$ matrix. A scalar λ is called an eigen value of A if $\exists$ (there exists) a non-zero vector x such that $Ax = \lambda x$. Such vector x is called an eigenvector of A corresponding to λ.

Note:

To find the eigenvalues, we find the values of λ which satisfies the characteristics equation of matrix A which is given by

$$\det\underbrace{(A - \lambda I)}_{\text{characteristic polynomial}} = 0 .$$

Example:- Find the eigenvalues and eigenvectors of the matrix A given by;

$$A = \begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix}$$

$$\det \left[\begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} \right]$$

$$\begin{vmatrix} 1-\lambda & -3 & 3 \\ 3 & -5-\lambda & 3 \\ 6 & -6 & 4-\lambda \end{vmatrix} = 0$$

$\Rightarrow (1-\lambda)[(-5-\lambda)(4-\lambda)-(-18)] - (-3)[3(4-\lambda)-(18)] + 3[-18-(-5-\lambda)6]$

$\Rightarrow (1-\lambda)[-20+5\lambda-4\lambda+\lambda^2+18] + 3[12-3\lambda-18] + [-54-(-30-6\lambda)]$

$\Rightarrow (1-\lambda)[\lambda+\lambda^2-2] + 3[-3\lambda-6] + 3[12-6\lambda]$

$\Rightarrow \lambda^2+\lambda-2-\lambda^3-\lambda^2+2\lambda-9\lambda-18+36+18\lambda = 0$

$\Rightarrow -\lambda^3+12\lambda+16=0$

$\Rightarrow \lambda^3-12\lambda-16=0$

Using -2; equation goes to zero.

$\lambda = -2$ satisfies the equation

$\lambda-(-2)$ is a factor of $\lambda^3-12\lambda-16$

$$\begin{array}{r|l} & \lambda^2-2\lambda-8 \\ \hline \lambda+2 & \lambda^3-12\lambda-16 \\ - & \lambda^3+2\lambda^2 \\ \hline & -2\lambda^2-12\lambda-16 \\ - & -2\lambda^2-4\lambda \\ \hline & -8\lambda-16 \\ - & -8\lambda-16 \\ \hline & 0 \quad 0 \end{array}$$

$\therefore \lambda^3-12\lambda-16 = (\lambda+2)(\lambda^2-2\lambda-8)$

$(\lambda+2)(\lambda^2-4\lambda+2\lambda-8)=0$

$(\lambda+2)(\lambda(\lambda-4)+2(\lambda-4))=0$

$(\lambda+2)(\lambda+2)(\lambda-4)=0$

$\lambda=4$, $\lambda=-2$ twice

05/11/2019.

Once the eigen values of A have been found, the eigen vectors can be found by Gaussian field method for eigen value λ, we have

$$(A - \lambda I)x = 0$$

So, we construct the augmented matrix and convert it to row echelon form

$$\begin{pmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{pmatrix}$$

$$\begin{pmatrix} 1-\lambda & -3 & 3 \\ 3 & -5-\lambda & 3 \\ 6 & -6 & 4-\lambda \end{pmatrix} = 0$$

using $\lambda = 4$.

$$\begin{pmatrix} -3 & -3 & 3 \\ 3 & -9 & 3 \\ 6 & -6 & 0 \end{pmatrix} = 0$$

Augmented matrix =

$$\begin{pmatrix} -3 & -3 & 3 & 0 \\ 3 & -9 & 3 & 0 \\ 6 & -6 & 0 & 0 \end{pmatrix}$$

$-\frac{1}{3}R_1 \rightarrow R_1$ $\begin{pmatrix} 1 & 1 & -1 & 0 \\ 3 & -9 & 3 & 0 \\ 6 & -6 & 0 & 0 \end{pmatrix}$

$\begin{array}{l} \\ R_2 - 3R_1 \rightarrow R_2 \\ R_3 - 6R_1 \rightarrow R_3 \end{array} \begin{pmatrix} 1 & 1 & -1 & 0 \\ 0 & -12 & 6 & 0 \\ 0 & -12 & 6 & 0 \end{pmatrix} \Longrightarrow \begin{array}{l} \\ R_3 - R_2 \\ \rightarrow R_3 \end{array} \begin{pmatrix} 1 & 1 & -1 & 0 \\ 0 & -12 & 6 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

$-\frac{1}{12}R_2 \begin{pmatrix} 1 & 1 & -1 & 0 \\ 0 & 1 & -\frac{1}{2} & 0 \\ 0 & -12 & 6 & 0 \end{pmatrix} \longrightarrow \begin{array}{l} \\ 12R_2 + R_3 \\ \rightarrow R_3 \end{array} \begin{pmatrix} 1 & 1 & -1 & 0 \\ 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

$R_2, \quad x_2 - \frac{1}{2}x_3 = 0$

$x_2 = \frac{1}{2}x_3$

$R_1, \quad x_1 + x_2 - x_3 = 0$

$x_1 + \frac{1}{2}x_3 - x_3 = 0$

$x_1 = x_3 - \frac{1}{2}x_3$

$x_1 = \frac{1}{2}x_3$

$X = \begin{pmatrix} x_1 = \frac{1}{2}x_3 \\ x_2 = \frac{1}{2}x_3 \\ x_3 = x_3 \end{pmatrix}$ $\quad X = x_3 \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 1 \end{pmatrix}$

This is the eigen value corresponding to $\lambda = 4$

Case 2, $\lambda = -2$

Augmented Matrix

$$\begin{pmatrix} 1-\lambda & -3 & 3 & 0 \\ 3 & -5-\lambda & 3 & 0 \\ 6 & -6 & 4-\lambda & 0 \end{pmatrix}$$

$$= \frac{1}{3}\begin{pmatrix} 3 & -3 & 3 & 0 \\ 3 & -3 & 3 & 0 \\ 6 & -6 & 6 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -1 & 1 & 0 \\ 3 & -3 & 3 & 0 \\ 6 & -6 & 6 & 0 \end{pmatrix} \begin{matrix} \\ R_2 - 3R_1 \longrightarrow R_2 \\ R_3 - 6R_1 \longrightarrow R_3 \end{matrix} \begin{pmatrix} 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

from row 1

$x_1 - x_2 + x_3 = 0$

$x_1 = x_2 - x_3$

$$X = \begin{pmatrix} x_1 = x_2 - x_3 \\ x_2 = \\ x_3 \end{pmatrix}$$

$$= x_2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

Assignment

Find the eigen values and eigen vectors of the following matrix

$$A = \begin{pmatrix} 2 & 4 & -3 \\ 0 & 2 & 0 \\ 0 & 3 & -1 \end{pmatrix}$$

$R_2: x_2 = 0$

$R_1: x_1 + 4/3 x_2 - x_3 = 0$

$x_1 = x_3$

$$X = \begin{pmatrix} x_1 = x_3 \\ x_2 = 0 \\ x_3 \end{pmatrix}$$

Solution:

$|A - \lambda I| = 0$

$$\begin{pmatrix} 2 & 4 & -3 \\ 0 & 2 & 0 \\ 0 & 3 & -1 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} = \begin{pmatrix} 2-\lambda & 4 & -3 \\ 0 & 2-\lambda & 0 \\ 0 & 3 & -1-\lambda \end{pmatrix}$$

$$\begin{vmatrix} 2-\lambda & 4 & -3 \\ 0 & 2-\lambda & 0 \\ 0 & 3 & -1-\lambda \end{vmatrix} \quad (2-\lambda)[(2-\lambda)(1-\lambda) - 0] - 4[0] - 3[0]$$

$= (2-\lambda)[2 - 2\lambda - \lambda + \lambda^2] = (2-\lambda)[2 - 3\lambda + \lambda^2]$

$= 4 - 6\lambda + 2\lambda^2 - 2\lambda + 3\lambda^2 - \lambda^3 = 4 - 8\lambda + 5\lambda^2 - \lambda^3$

using $\lambda = -1$, everything goes to 0 $\therefore$ $\lambda + 1$ is a factor

$(\lambda+1)(\lambda-2)(\lambda-2)$

$\lambda = -1$ and 2 twice

$\therefore$ we have $(A - \lambda I)x = 0$

$$\begin{pmatrix} 2-\lambda & 4 & -3 \\ 0 & 2-\lambda & 0 \\ 0 & 3 & -1-\lambda \end{pmatrix} \text{ using } \lambda = -1 \quad \begin{pmatrix} 3 & 4 & -3 \\ 0 & 3 & 0 \\ 0 & 3 & 0 \end{pmatrix}$$

Augmented matrix

$$\begin{pmatrix} 3 & 4 & -3 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 3 & 0 & 0 \end{pmatrix} \begin{matrix} 1/3 R_1 \\ R_3 - R_2 \to R_3 \end{matrix} \begin{pmatrix} 1 & 4/3 & -1 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 3 & 0 & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 4/3 & -1 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 4/3 & -1 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 3 & 0 & 0 \end{pmatrix} \begin{matrix} 1/3 R_2 \Rightarrow \\ R_3 - 1/3 R_2 \end{matrix} \begin{pmatrix} 1 & 4/3 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 3 & 0 & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 4/3 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Similar Matrices

12/11/2019

Definition:- Let A and B be n×n matrices. then, we say that A is similar to B if there exists an invertible n×n matrix P such that $A = PBP^{-1}$

NOTE:- A is similar to A if $A = PAP^{-1}$.

Also, if A is similar to B such that $A = PBP^{-1} \Rightarrow PAP^{-1} = B$

$PAP^{-1} = B$

suppose A is similar to B, such that $A = PBP^{-1}$

suppose B is similar to C, $B = TCT^{-1}$

$$A = P(TCT^{-1})P^{-1} = (PT)C(PT)^{-1}$$

$\Rightarrow$ A is similar to C $\Rightarrow$ Transitivity.

If n×n matrix A and B are similar, they have ~~the~~ the same characteristic polynomial and hence they have the same eigen value

NOTE:- ~~Example~~ The necessary but not sufficient condition for two matrices to be similar is that their determinant must be equal.

Example:- Given A & B. show if A and B are similar or not.

(i) $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 4 \\ 4 & 5 \end{bmatrix}$ $|A| \neq |B|$; hence they are not similar

(2) $A = \begin{bmatrix} -2 & 6 \\ -2 & 5 \end{bmatrix}$ $B = \begin{bmatrix} -4 & 3 \\ -10 & 7 \end{bmatrix}$ $|A| = |B|$

let $P = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$A = PBP^{-1} \Rightarrow AP = PB$

$$\begin{bmatrix} -2 & 6 \\ -2 & 5 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} -4 & 3 \\ -10 & 7 \end{bmatrix}$$

$-2a + 6c = -4a - 10b$ —— i

$-2b + 6d = 3a + 7b$ —— ii

$-2a + 5c = -4c - 10d$ —— iii

$-2b + 5d = 3c + 7d$ —— iv

from (i)

$a = -3c - 5b$

using a in (iii)

$d = -3/2\, c - b$

$$\begin{bmatrix} -2 & 6 \\ -2 & 5 \end{bmatrix}\begin{bmatrix} -5 & 1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} -5 & 1 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} -4 & 3 \\ -10 & 7 \end{bmatrix}$$

$AP = PB = \begin{bmatrix} 10 & -8 \\ 10 & -7 \end{bmatrix}$

A matrix A is said to be diagonizable if it is similar to a diagonal matrix D

$$A = PDP^{-1}$$

Theory:

An $n \times n$ matrix is diagonalizable iff it has the n linearly independent eigen values

A is similar to a diagonal matrix D whose diagonal elements are the eigen values i.e $A = PDP^{-1}$ where P is the matrix whose columns are the n linearly independent eigen values where D and P are found in a given matrix A has been diagonalized.

Example:

Diagonalize the following

1. $A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 1 \\ -1 & 0 & 1 \end{bmatrix}$ 2. $B = \begin{bmatrix} -1 & 0 & 2 \\ 0 & 2 & 0 \\ 3 & 0 & 4 \end{bmatrix}$

step I: find the eigen values

II: find the eigen vectors

III: form the matrix P of the eigen vectors

$V_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ $V_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ $V_3 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$$P = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$